SCATTERING AMPLITUDES IN THEORIES OF COMPACTIFIED GRAVITY

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TABLE OF CONTENTS

Chapter 1

Introduction

High-energy physics is the study of fundamental particles and their interactions. The success of modern high-energy physics is owed to the hard work of many experimental and theoretical physicists, including their development and application of quantum field theories. A quantum field theory (QFT) models each fundamental particle as an excitation of a field corresponding to that particle's species. Relativistic QFTs in particular combine the universal speed of light from special relativity (which provides well-defined meanings of particle mass and spin) with the probabilistic nature of reality that is intrinsic to quantum mechanics. With the help of a few additional features (the cluster decomposition principle, the LSZ reduction formula, etc.), high-energy physicists can calculate the probability that certain combinations of particles become other combinations of particles via scattering processes; knowing these probabilities allows the calculation of experimentally-relevant cross-sections and decay rates. However, before these probabilities can be calculated, the interested physicist must first calculate the Lorentz-invariant matrix element corresponding to the relevant scattering process, and to do that a physicist requires a Lagrangian.[1]

Modern quantum field theory has streamlined the construction of model Lagrangians. In essence, a physicist decides on what matter particles and forces they would like included, chooses some interesting processes to investigate, and then puts together a Lagrangian that sums all terms consistent with that content which are relevant to those processes. Forces are typically included by declaring that the Lagrangian should have certain local symmetries, which then generate gauge bosons and their couplings to the matter particles. This is the way in which the champion of modern high-energy physics–the Standard Model (SM)–is constructed. The SM is presently our most accurate description of reality at subatomic scales, with high-energy experiments repeatedly confirming its predictions to increasingly high precision.

Prior to electroweak symmetry breaking (more on that in a moment), the Standard Model is an $\mathbf{SU(3)_C} \times \mathbf{SU(2)_W} \times \mathbf{U(1)_Y}$ gauge theory where

- $\mathbf{SU(3)_C}$ generates the strong interaction and is gauged by eight gluons G^a_μ,
- $\mathbf{SU(2)_W}$ generates the weak isospin interaction and is gauged by the triplet of vector bosons $\{W^1_\mu, W^2_\mu, W^3_\mu\}$, and
- $\mathbf{U(1)_Y}$ generates the weak hypercharge interaction and is gauged by the vector boson B_μ.

[1] We follow the standard high-energy convention of calling what is actually a "Lagrangian density" (the integrand of an integral over spacetime) simply a "Lagrangian" (which would otherwise be the integrand of an integral over time).

The Matter Content of the Pre-EWSB Standard Model

	Field Symbol		Mass	Spin	$\mathbf{U(1)_Y}$	$\mathbf{SU(2)_W}$	$\mathbf{SU(3)_C}$
Left-Handed Quarks	$\mathbf{q_{mL}}$	$\mathbf{u_{mL}}$	0	$\frac{1}{2}$	$+\frac{1}{3}$	$+\frac{1}{2}$	triplet
		$\mathbf{d_{mL}}$				$-\frac{1}{2}$	
Left-Handed Leptons	$\ell_{\mathbf{mL}}$	$\nu_{\mathbf{mL}}$	0	$\frac{1}{2}$	-1	$+\frac{1}{2}$	singlet
		$\mathbf{e_{mL}}$				$-\frac{1}{2}$	
Right-Handed Quarks		$\mathbf{u_{mR}}$	0	$\frac{1}{2}$	$+\frac{4}{3}$	0	triplet
		$\mathbf{d_{mR}}$	0	$\frac{1}{2}$	$-\frac{2}{3}$	0	triplet
Right-Handed Leptons		$\nu_{\mathbf{mR}}$	0	$\frac{1}{2}$	0	0	singlet
		$\mathbf{e_{mR}}$	0	$\frac{1}{2}$	-2	0	singlet
Higgs Doublet	$\mathbf{\Phi}$	ϕ^+	0	0^*	+1	$+\frac{1}{2}$	singlet
		$\phi^{\mathbf{0}}$				$-\frac{1}{2}$	

Table 1.1: The matter content of the Standard Model prior to electroweak symmetry breaking (EWSB) including their masses, internal spins, and gauge transformation properties. Rows are organized as to indicate matter fields that are related by the weak gauge group $\mathrm{SU(2)_W}$, i.e. q_{mL} labels a weak gauge doublet with $+1/2$ component u_{mL} and $-1/2$ component d_{mL}. The index $m \in \{1, 2, 3\}$ labels the generation of a given quark (q, u, d) or lepton (ℓ, e, ν) field, while a subscript "L" or "R" indicates whether it has left or right-handed chirality. The pre-EWSB Standard Model also contains gauge bosons B_μ, $\{W^1_\mu, W^2_\mu, W^3_\mu\}$, and $\{G^1_\mu, \ldots, G^8_\mu\}$ corresponding to the weak hypercharge $\mathrm{U(1)_Y}$, weak isospin $\mathrm{SU(2)_W}$, and strong $\mathrm{SU(3)_C}$ gauge groups respectively. The left- and right-handed neutrinos ν_{mL} and ν_{mR} are called active and inert neutrinos respectively based on their $\mathrm{SU(2)_W}$ transformation properties (or lack thereof). Whether or not the inert neutrinos ν_{mR} exist is an open question.

The matter content of the theory (including each particle's mass, spin, and transformation behavior under the aforementioned local symmetry groups) is listed in Table 1.1. The spin-$\frac{1}{2}$ quarks and leptons exhibit a generational structure (as emphasized by the subscript $m \in \{1, 2, 3\}$ on each field), the spin-0 Higgs doublet Φ does not, and all particles are massless. Everything changes when the electroweak gauge group $\mathbf{SU(2)_W \times U(1)_Y}$ becomes spontaneously broken [1, 2, 3].

The electroweak gauge group breaks because the Higgs doublet spontaneously acquires a vacuum expectation value (vev), $v_{\mathrm{EW}} = 0.246$ TeV, thereby isolating the Higgs boson H from the rest of the doublet at low energies. This causes the electroweak gauge groups $\mathbf{SU(2)_W \times U(1)_Y}$ to spontaneously break down to the electromagnetic gauge group $\mathbf{U(1)_Q}$. When this happens, a superposition of the W^3_μ and B_μ bosons become the massless spin-1 photon A_μ that gauges $\mathbf{U(1)_Q}$, while (in unitary gauge) an orthogonal mixture absorbs a fraction of the remaining Higgs doublet and becomes the massive Z-boson Z_μ. The other $\mathbf{SU(2)_W}$ gauge bosons W^1_μ and W^2_μ absorb the rest of the Higgs doublet to become the massive W-bosons $W^\pm_\mu$. Simultaneously, interactions between the Higgs doublet and the (massless) fermionic matter fields are turned into mass and mixing terms, ultimately resulting in newly massive fermionic matter. Overall, electroweak symmetry breaking causes the low-energy SM to become an $\mathbf{SU(3)_C} \times \mathbf{U(1)_Q}$ gauge theory, wherein

- $\mathbf{SU(3)_C}$ still generates the strong interaction and is gauged by the gluons G^a_μ, and
- $\mathbf{U(1)_Q}$ generates the electromagnetic interaction and is gauged by the photon A_μ.

and the matter content is as listed in Table 1.2. In this way, electroweak symmetry breaking simultaneously explains the masses of the electroweak gauge bosons, expresses the weak force in terms of a local symmetry group, and generates masses for the Standard Model matter particles. The possibility that a single mechanism ("the Higgs mechanism") could explain all of these features motivated physicists in the 1960's to hypothesize the existence of the Higgs boson [4, 5, 6]. Its eventual experimental confirmation in 2012 by the ATLAS and CMS collaborations at CERN is among the most celebrated achievements of physics in the 21st century thus far [7, 8].

The SM is so successful in its predictions of subatomic phenomena that nearly every physically-descriptive QFT investigated in the high-energy literature hypothesizes new particles simply as add-ons to the SM. Of course, despite all that the Standard Model can predict, many physical phenomena lie outside its reach. For example, the SM does not predict the natures of neutrinos or dark matter or dark energy, nor does it incorporate gravity.

A limited version of gravity *can* be added to the SM by considering four-dimensional general relativity in the weak field limit. Doing so generates a particle description of gravity, wherein the gravitational force is mediated by a massless spin-2 particle called the graviton. However, this modification breaks down at the Planck scale (or mass) $M_{\text{Pl}} = 2.435 \times 10^{15}$ TeV, reflecting its inability to describe strong or intrinsically quantum gravitational phenomenon that occur at higher energies. Furthermore, this SM + gravity theory possesses a vast range of energy scales between the electroweak's v_{EW} and gravity's M_{Pl} across which there is no new physics. Although nothing prevents such a hierarchy of scales in principle, the large ratio between the energy scales $M_{\text{Pl}}/v_{\text{EW}} \sim 5 \times 10^{16}$ is technically unnatural.[2]

For many decades, physicists have attempted to solve this "hierarchy problem" by hypothesizing a physical mechanism that would naturally generate a large ratio of scales $M_{\text{Pl}}/v_{\text{EW}}$. For example, in 1999, Randall and Sundrum proposed a five-dimensional gravity theory that could reparameterize the hierarchy problem via the warping of a non-factorizable extra-dimensional spacetime geometry [10, 11]. This theory is the Randall-Sundrum 1 (RS1) model, and the focus of this dissertation.

[2]Naturalness has several technical definitions in high-energy physics, but a theory tends to be natural if all of its parameters are set to values with similar magnitudes. Because the Higgs boson is a scalar particle, its mass-squared m_H^2 receives quantum corrections proportional to the square of the largest scales in the theory. By including gravity, that largest scale is the Planck mass M_{Pl}, and $16^2 = 256$ decimal places of cancellations are required to obtain the experimentally-measured mass $m_H = 125$ GeV $\sim v_{\text{EV}}$ instead of a Planck scale mass $m_H \sim M_{\text{Pl}}$. Thus, the theory parameters must be fine-tuned to ensure this cancellation, and the ratio $M_{\text{Pl}}/v_{\text{EH}}$ is technically unnatural in the Standard Model.

The Matter Content of the Post-EWSB Standard Model

	Name	Symbol	Mass (GeV/c^2)	Spin	$\mathrm{SU(3)_C}$	$\mathrm{U(1)_Q}$
Up-Type Quarks	up quark	u	2.3×10^{-3}	$\frac{1}{2} \otimes \frac{1}{2}$	triplet	$+\frac{2}{3}$
	charm quark	c	1.28			
	top	t	173			
Down-Type Quarks	down quark	d	4.7×10^{-3}	$\frac{1}{2} \otimes \frac{1}{2}$	triplet	$-\frac{1}{3}$
	strange quark	s	9.5×10^{-2}			
	bottom quark	b	4.18			
Neutral Leptons	neutrino 1	ν_1	?	?	singlet	0
	neutrino 2	ν_2	?			
	neutrino 3	ν_3	?			
Charged Leptons	electron	e	5.11×10^{-4}	$\frac{1}{2} \otimes \frac{1}{2}$	singlet	-1
	muon	μ	0.106			
	tauon	τ	1.78			
	Higgs boson	H	125	0	singlet	0
	Z boson	Z	91.2	1	singlet	0
	W boson(s)	W^+	80.3	1^*	singlet	$+1$

Table 1.2: The matter content of the Standard Model after electroweak symmetry breaking (EWSB) including their masses, internal spins, and gauge transformation properties [9]. Rows group together matter fields that are related by generational structure. The Standard Model also contains the photon A_μ and the gluons $\{G^1_\mu, \ldots, G^8_\mu\}$ which are the gauge bosons corresponding to the electromagnetic $\mathrm{U(1)_Q}$ and strong $\mathrm{SU(3)_C}$ gauge groups respectively. The precise nature of the masses and spin structure of the neutrinos is an open question. The neutrino mass eigenstates ν_1, ν_2, ν_3 are often reorganized via superposition into weak isospin eigenstates ν_e, ν_μ, ν_τ called the electron, muon, and tauon neutrinos respectively, which reconstruct the pre-EWSB active neutrinos at the cost of no longer having definite mass.

Relative to the usual four-dimensional (4D) spacetime, the RS1 model adds a finite extra dimension of space with length πr_cwhich is parameterized by a coordinate $y \in \{0, \pi r_c\}$, where r_c is called the compactification radius. At either end of the dimension is a four-dimensional hypersurface called a brane, with the five-dimensional spacetime between the branes being called the bulk. Typically, the four-dimensional world as we know it (e.g. the matter content) is placed on one brane (the "visible brane") and only gravity is allowed to freely propagate through the bulk. Extra-dimensional warping is achieved by the presence of a warp factor $\varepsilon \equiv e^{-kr_c|\varphi|}$ in the RS1 spacetime metric, where k is called the warping parameter and $\varphi \equiv y/r_c \in \{0, \pi\}$ is a unitless version of the extra-dimensional coordinate. This warp factor enters into other aspects of RS1 calculations. For example, a fundamental energy scale Λ in the bulk can be warped down to $\Lambda\, e^{-kr_c\pi}$ for an observer on the visible brane. In particular, we can set $\Lambda \approx M_{\text{Pl}}$ and its warped value $\Lambda\, e^{-kr_c\pi} \approx v_{\text{EW}}$ by choosing $kr_c \approx 12$, such that the hierarchy problem has gone from trying to explain the large ratio $M_{\text{Pl}}/v_{\text{EW}} \sim 5 \times 10^{16}$ to trying to explain the order-10 number $kr_c \sim 12$. Unfortunately, this warp factor is not universally beneficial: whereas strong & quantum gravitational effects force 4D gravity to break down at M_{Pl}, the RS1 model breaks down at the scale $\Lambda_\pi \equiv M_{\text{Pl}}\, e^{-kr_c\pi}$ instead. Thus, if $kr_c \sim 12$ as motivated by the hierarchy problem, then $\Lambda_\pi \sim v_{\text{EW}}$, and the theory becomes strongly coupled at LHC-relevant energy scales. As collider constraints confirm the Standard Model to increasingly high energies, kr_c is driven necessarily lower, and the RS1 models creeps further away from a solution to the hierarchy problem. Nowadays, the RS1 model is utilized in relation to theoretical problems such as the AdS/CFT correspondence [12, 13] and as a model that generates phenomenologically-interesting massive spin-2 particles [14].

Regardless of the specific value of kr_c used, the size πr_c of the extra-dimension is assumed small so that the five-dimensional (5D) nature of spacetime remains hidden at low energies (thereby explaining why we do not experience an extra spatial dimension in everyday life). In a sense, the relationship between the 5D RS1 spacetime and the usual 4D spacetime is similar to the relationship between a realistic sheet of paper (which has small but finite thickness) and its approximation as a two-dimensional plane. Because particles with sufficient energy can propagate throughout the full five-dimensional RS1 spacetime, the symmetry group relevant to high-energy particles is the 5D RS1 diffeomorphism group, which is gauged by the 5D RS1 graviton described by a 5D field $\hat{H}(x, y)$. At low energies, particles can no longer meaningfully probe the extra dimension, and the 5D RS1 diffeomorphism group is spontaneously broken down to a subgroup containing the usual 4D diffeomorphism group, which is gauged by the 4D graviton described by a 4D field $\hat{h}^{(0)}(x)$. In total, spontaneous symmetry breaking in the RS1 model results in the following 4D particle content:

- the 4D graviton, $h^{(0)}$, a massless spin-2 particle
- the radion, $r^{(0)}$, a massless spin-0 particle
- KK modes, $h^{(n)}$ for $n \in \{1, 2, \dots\}$, infinitely many massive spin-2 particles

in a process called Kaluza-Klein (KK) decomposition. The value n for a particular KK mode $h^{(n)}$ is called its KK number. The KK modes gain masses by absorbing degrees of freedom from the 5D RS1 graviton, which is reflected in the fact that a massive spin-2 particle in four dimensions and a massless 5D graviton both have five states. Because of its qualitative similarities to electroweak symmetry breaking and its use of a nontrivial background geometry to achieve spontaneous symmetry breaking, this has been referred to as a “geometric Higgs mechanism” [15]. The radion $r^{(0)}$ is a massless spin-0 particle generated by disturbing the separation distance between the branes.

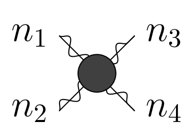
Due to their common origin in the RS1 model, the scattering of 4D gravitons and the scattering of massive KK modes are closely related. In particular, (as demonstrated in this dissertation) the high-energy growth of the matrix elements describing 4D graviton and KK mode scatterings are identical. Before describing how this is possible in the RS1 model, let us first describe an analogous calculation in a model with finitely many particles: the Standard Model. In this case, the intermediate vector bosons ($W^{\pm}$, Z) are special with respect to electroweak symmetry breaking (EWSB) because they are massive superpositions of the original $\mathbf{SU(2)_W} \times \mathbf{U(1)_Y}$ gauge bosons (W^1,W^2,W^3,B); this contrasts with the situation of the fermions and even the Higgs boson, although they also gain masses as a result of EWSB. The only superposition of $\mathbf{SU(2)_W} \times \mathbf{U(1)_Y}$ gauge bosons that remains massless is the photon (γ), which gauges the electromagnetic $\mathbf{U(1)_Q}$.

Because the photon has no cubic or quartic self-interactions, its center-of-momentum frame 2-to-2 tree-level scattering matrix element (hereafter referred to simply as “matrix element” for brevity) vanishes identically: $\mathcal{M} = 0$. Let E denote the incoming center-of-momentum energy of this process, so that the Mandelstam variable s equals E^2. In terms of high-energy growth, the photon scattering matrix element (trivially) scales like $\mathcal{O}(s^0)$. Another way in which we could have arrived at this same scaling is by combining the following facts:

- A 4D matrix element must be unitless.
- There is no energy scale available to this process.

The latter point means that there are no quantities with which to cancel any powers of energy introduced by factors of s, and thus the only way for the matrix element to be consistent with the first point is to scale like $\mathcal{O}(s^0)$ at high energies (which $\mathcal{M} = 0$ does trivially, as previously mentioned). Diagrammatically, we write

$$\mathcal{M}_{\gamma\gamma\to\gamma\gamma} =$$



$$\sim \qquad \mathcal{O}(s^0) \tag{1.1}$$

In contrast, the 2-to-2 scattering of massive spin-1 particles such as the W-bosons *does* have access to another energy scale: the particle’s mass. For example, an external massive spin-1 particle with mass m, 4-momentum p, and helicity λ will enter a matrix element calculation with any one of three possible polarization vectors:

$$[\epsilon^\mu_{\pm1}(p)] = \pm\frac{e^{\pm i\phi}}{\sqrt{2}}\begin{pmatrix} 0 \\ -c_\theta c_\phi \pm i s_\phi \\ -c_\theta s_\phi \mp i c_\phi \\ s_\theta \end{pmatrix} \qquad [\epsilon^\mu_0(p)] = \frac{1}{m}\begin{pmatrix} |\vec{p}| \\ E\, c_\phi s_\theta \\ E\, s_\phi s_\theta \\ E\, c_\theta \end{pmatrix} = \frac{1}{m}\begin{pmatrix} |\vec{p}| \\ E\,\hat{p} \end{pmatrix} \tag{1.2}$$

corresponding to helicities $\lambda = \pm 1$ and $\lambda = 0$ respectively, where (ϕ, θ) determines the 3-direction of $\vec{p}$ in spherical coordinates and $(c_x, s_x) \equiv (\cos x, \sin x)$. The components of the helicity-zero polarization vector $\epsilon_0^\mu(p)$ diverge like $\mathcal{O}(E/m) = \mathcal{O}(\sqrt{s}/m)$ at high energies, which is only made possible by the existence of the mass m. A massless spin-1 particle such as the photon only has access to the helicity $\lambda = \pm 1$ states, which are independent of mass and energy.

Because each massive spin-1 state has three helicity options, the external states in a 2-to-2 massive spin-1 scattering process can be in any one of $3^4 = 81$ helicity combinations (although many of these are related to one another through crossing symmetry). Because the helicity-zero polarization vector diverges most quickly in energy, it is perhaps unsurprising that the fastest growing matrix element is typically attained by setting all external helicities to zero. We will refer to such a process as a "helicity-zero process." It is not unusual for a helicity-zero matrix element describing massive spin-1 scattering to grow as fast as $\mathcal{O}(s^2)$ at high energies.

However, this is not what happens in the SM. Instead, the helicity-zero matrix grows like $\mathcal{O}(s^0)$:

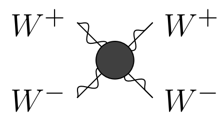
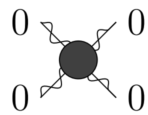
$\mathcal{M}_{WW\to WW} =$  $\overset{\text{helicity}}{\underset{\text{zero}}{\sim}} \quad \mathcal{O}(s^0)$ (1.3)

Table 1.3 summarizes the various diagrams that sum to form this matrix element, including their individual high-energy behaviors. Several channels exhibit $\mathcal{O}(s^2)$ growth, but cancellations occur when all diagrams are summed together which ultimately result in a net $\mathcal{O}(s^0)$ growth just like the photon scattering matrix element.

The cancellations which reduce $\mathcal{O}(s^2)$ growth to $\mathcal{O}(s^0)$ growth are not a coincidence: even though the electroweak gauge group $\mathbf{SU(2)_W} \times \mathbf{U(1)_Y}$ has been spontaneously broken down to the electromagnetic gauge group $\mathbf{U(1)_Q}$, this fundamental symmetry still protects the scattering behavior of the related gauge bosons. Thus, the overall high-energy growth of the matrix element describing 2-to-2 scattering of the W-bosons (which are superpositions of the $\mathbf{SU(2)_W}$ gauge bosons) matches that of the 2-to-2 scattering of photons (which gauge the remaining $\mathbf{U(1)_Q}$).

The main result of this dissertation is the demonstration that similar cancellations occur in the Randall-Sundrum 1 model. In this case, a nontrivial background geometry at low energies causes the 5D RS1 diffeomorphism group to be spontaneously broken down to a subgroup containing the 4D diffeomorphism group. This latter group is gauged by the usual massless graviton.

Unlike the case of photon scattering that we previously considered, 4D gravity has an implicit energy scale: the Planck mass M_{Pl}. This scale enters the graviton scattering matrix element via the 4D gravitational coupling $\kappa_{\text{4D}} \equiv 2/M_{\text{Pl}}$, of which two instances are present in any given tree-level diagram. In order to be unitless overall, the matrix element must contribute a factor of $s = E^2$ to compensate, and thus it grows like

$\mathcal{M}_{00\to 00} =$  $\sim \quad \mathcal{O}(s)$ (1.4)

at high energies. The label "0" indicates the 4D graviton, $h^{(0)}$, each instance of which can have helicity $\lambda = \pm 2$.

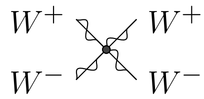
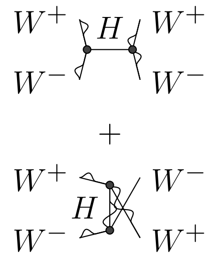
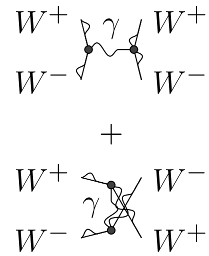
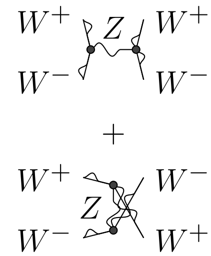
$\mathcal{M}_{WW\to WW} =$	$\mathcal{M}_{\rm c}$	$+\mathcal{M}_H$	$+\mathcal{M}_\gamma$	$+\mathcal{M}_Z$
Mediator:	-	Higgs	photon	Z-boson
Diagrams:				
Helicity-Zero High-Energy Scaling:	$\sim \mathcal{O}(s^2)$	$\sim \mathcal{O}(s)$	$\sim \mathcal{O}(s^2)$	$\sim \mathcal{O}(s^2)$

Table 1.3: The various diagrams that contribute to the tree-level matrix element for the 2-to-2 Standard Model scattering process $W^+W^- \to W^+W^-$ and their high-energy behaviors. The tree-level matrix element $\mathcal{M}_{WW\to WW}$ from Eq. (1.3) is the sum of these diagrams. Because of cancellations between diagrams, $\mathcal{M}_{WW\to WW}$ scales like $\mathcal{O}(s^0)$, just like the 2-to-2 photon scattering matrix element $\mathcal{M}_{\gamma\gamma\to\gamma\gamma}$.

If we instead consider tree-level 2-to-2 scattering of massive spin-2 particles (such as the RS1 KK modes), then each external state will be associated with any one of five possible polarization tensors, $\epsilon^{\mu\nu}_\lambda(p)$:

$$\epsilon^{\mu\nu}_{\pm 2}(p) = \epsilon^{\mu}_{\pm 1}(p)\,\epsilon^{\nu}_{\pm 1}(p)\ , \tag{1.5}$$

$$\epsilon^{\mu\nu}_{\pm 1}(p) = \frac{1}{\sqrt{2}}\left[\epsilon^{\mu}_{\pm 1}(p)\,\epsilon^{\nu}_{0}(p) + \epsilon^{\mu}_{0}(p)\,\epsilon^{\nu}_{\pm 1}(p)\right] \tag{1.6}$$

$$\epsilon^{\mu\nu}_{0}(p) = \frac{1}{\sqrt{6}}\left[\epsilon^{\mu}_{+1}(p)\,\epsilon^{\nu}_{-1}(p) + \epsilon^{\mu}_{-1}(p)\,\epsilon^{\nu}_{+1}(p) + 2\epsilon^{\mu}_{0}(p)\,\epsilon^{\nu}_{0}(p)\right]\ , \tag{1.7}$$

where $\epsilon^{\mu}_\lambda(p)$ are the previously-defined spin-1 polarization vectors. As in the massive spin-1 case, the most divergent of these is the helicity-zero option, which grows like $\mathcal{O}(s/m^2)$ at large energies. Massive spin-2 scattering matrix elements have $5^4 = 625$ possible helicity combinations (many related to one another via crossing symmetry), but the helicity-zero combination is typically the most divergent, usually growing as fast as $\mathcal{O}(s^5)$.

Keeping this in mind, consider the matrix element $\mathcal{M}_{n_1 n_2\to n_3 n_4}$ corresponding to the helicity-zero KK mode scattering process $h^{(n_1)}h^{(n_2)} \to h^{(n_3)}h^{(n_4)}$ where the KK numbers n_1, n_2, n_3, and n_4 are all nonzero. Table 1.4 summarizes the diagrams which sum to form $\mathcal{M}_{n_1 n_2\to n_3 n_4}$ and their high-energy behaviors. As anticipated in the previous paragraph, nearly every diagram that contributes to this matrix element diverges like $\mathcal{O}(s^5)$. However, this dissertation demonstrates explicitly that nontrivial cancellations occur between these

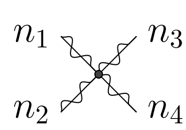
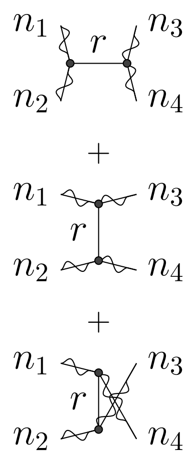
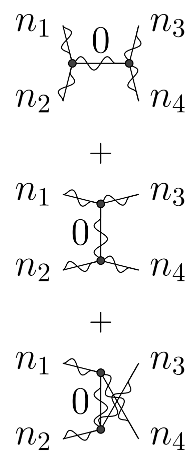
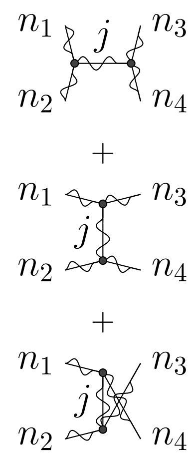
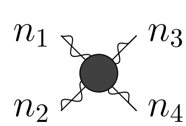
$\mathcal{M}_{n_1 n_2 \to n_3 n_4} =$	$\mathcal{M}_{\rm c}$	$+\mathcal{M}_{\rm r}$	$+\mathcal{M}_0$	$+\sum\limits_{j>0} \mathcal{M}_j$
Mediator:	-	radion	graviton	massive spin-2 KK mode
Diagrams:				
Helicity-Zero High-Energy Scaling:	$\sim \mathcal{O}(s^5)$	$\sim \mathcal{O}(s^3)$	$\sim \mathcal{O}(s^5)$	$\sim \mathcal{O}(s^5)$

Table 1.4: The various diagrams that contribute to the tree-level matrix element for the 2-to-2 RS1 model scattering process $h^{(n_1)}h^{(n_2)} \to h^{(n_3)}h^{(n_4)}$ and their high-energy behaviors. The tree-level matrix element $\mathcal{M}_{n_1 n_2 \to n_3 n_4}$ from Eq. (1.8) is the sum of these diagrams. Because of cancellations between diagrams, the overall matrix element $\mathcal{M}_{n_1 n_2 \to n_3 n_4}$ scales like $\mathcal{O}(s)$, just like the 2-to-2 graviton scattering matrix element $\mathcal{M}_{00\to 00}$. The confirmation and detailed demonstration of these cancellations is a major result of this dissertation.

infinitely-many diagrams such that the full matrix element diverges like $\mathcal{O}(s)$:

$$\mathcal{M}_{n_1 n_2 \to n_3 n_4} = \quad \text{[diagram]} \quad \underset{\text{zero}}{\overset{\text{helicity}}{\sim}} \quad \mathcal{O}(s) \tag{1.8}$$



which is precisely the energy growth found in the 4D graviton scattering channel. The conceptual similarities between the Standard Model and RS1 model are summarized in Table 1.5, with our original results indicated in red. We also demonstrate in this dissertation that the RS1 strong-coupling scale $\Lambda_\pi = M_{\rm Pl}\, e^{-krc\pi}$ can be calculated directly from the 4D effective RS1 model.

Additionally, in practice if we intend to perform a numerical calculation (as might be relevant to experimental applications of the RS1 model) then we must truncate the number of KK modes we include as intermediate states (e.g. replacing the sum $\sum_{j=0}^{+\infty} \mathcal{M}_j$ in the

	Standard Model	**Randall-Sundrum 1**
The fundamental symmetry group...	$\mathbf{SU(2)_W \times U(1)_Y}$	5D diffeomorphisms
... w/ unitarity-violation scale...	N/A	$\Lambda_\pi = M_{\rm Pl}\, e^{-krc\pi}$
... and gauged by the...	electroweak bosons	5D RS1 graviton
... is spontaneously broken by...	the Higgs vev	background geometry
... to a new symmetry group...	$\mathbf{U(1)_Q}$	4D diffeomorphisms*
... gauged by the...	photon, γ	4D graviton, $h^{(0)}$
... resulting in a spin-0 state...	Higgs boson, H	radion, $r^{(0)}$
... as well as massive states built from fund. gauge bosons...	W-bosons, $W^\pm$ and Z-boson, Z	spin-2 KK modes, $h^{(n)}$ for $n \in \{1, 2, \dots\}$
The 2-to-2 gauge boson process...	$\gamma\gamma \to \gamma\gamma$	$h^{(0)}h^{(0)} \to h^{(0)}h^{(0)}$
... has $\mathcal{M}$ w/ high-energy growth $\sim$	$\mathcal{O}(s^0)$	$\mathcal{O}(s)$
... or, if naively given mass, ...	$\mathcal{O}(s^2)$	$\mathcal{O}(s^5)$
... yet 2-to-2 massive state process where mass arises via sym. break...	$W^+W^- \to W^+W^-$	$h^{(n_1)}h^{(n_2)} \to h^{(n_3)}h^{(n_4)}$
... has $\mathcal{M}$ w/ high-energy growth $\sim$	$\mathcal{O}(s^0)$	$\mathbf{O(s)}$
Breaking the fund. symmetry by...	elim. Z	KK tower truncation
... makes massive states scatter like naively-massive gauge bosons, $\mathcal{M} \sim$	$\mathcal{O}(s^2)$	$\mathbf{O(s^5)}$
Breaking the fund. symmetry by...	elim. the Higgs	elim. the radion
... makes massive states scatter $\sim$	$\mathcal{O}(s)$	$\mathbf{O(s^3)}$

Table 1.5: The Standard Model (SM) and the Randall-Sundrum 1 (RS1) model share a chain of conceptual similarities with respect to the scattering of particles made massive by spontaneous symmetry breaking. The Mandelstam variable $s \equiv E^2$, where E is the incoming center-of-momentum energy. Original results presented in this dissertation are indicated in bold. (* - Technically, the new symmetry group is the Cartan subgroup of the 5D diffeomorphisms that contains the 4D diffeomorphisms.)

matrix element with $\sum_{j=0}^{N}\mathcal{M}_j$ for some integer N). Because the entire tower is required in order to cancel the leading $\mathcal{O}(s^5)$ growth, truncating the KK tower too low can cause the matrix element to violate partial wave unitarity well below the strong-coupling scale Λ_π. Furthermore, because the radion contributes matrix elements with $\mathcal{O}(s^3)$ growth, proper inclusion of the radion is also vital to avoiding partial wave unitarity constraints. The effect of KK tower truncation and inclusion of the radion on the accuracy of KK mode scattering matrix elements is also investigated in this dissertation.

The remainder of the dissertation details the original results published in [16, 17, 18], as well as generalizing and elaborating on aspects of those calculations in ways that have not yet been submitted for publication. It is organized as follows:

- Chapter 2 establishes definitions and conventions from 4D quantum field theory relevant to the dissertation. In the interest of acting as a useful resource, it also provides a detailed derivation of 2-to-2 partial wave unitarity constraints and helicity eigenstates from first principles.

- Chapter 3 calculates the 5D weak field expanded RS1 Lagrangian $\mathcal{L}_{\text{5D}}$ to quartic order in the 5D fields or (equivalently) second order in the 5D coupling κ_{5D}. We demonstrate that all terms containing factors of $(\partial_\varphi|\varphi|)$ or $(\partial_\varphi|\varphi|)$ are cancelled to all orders in κ_{5D}.

- Chapter 4 presents an original parameterization of the 4D effective RS1 Lagrangian which manifests as a "5D-to-4D formula" and categorizes all RS1 couplings as either "A-type" or "B-type" depending on the associated derivative content of the interaction. Many relationships between RS1 couplings and masses are derived; these significantly generalize our existing published work and will be submitted for publication in a future paper.

- Chapter 5 demonstrates that the matrix element describing massive spin-2 KK mode scattering in the 5D orbifolded torus and RS1 models exhibits $\mathcal{O}(s)$ growth after cancellations of more divergent behavior. From cancellations in the helicity-zero elastic case ($h^{(n)}h^{(n)} \to h^{(n)}h^{(n)}$) we derive sum rules relating KK mode masses and couplings, all but one of which we prove analytically. The final sum rule is demonstrated numerically. The RS1 strong-coupling scale $\Lambda_\pi = M_{\text{Pl}}\, e^{-kr_c\pi}$ is calculated numerically in the 4D effective RS1 model and the effect of KK tower truncation on matrix element accuracy is investigated. These important original results have been published across several papers [16, 17, 18].

- Chapter 6 concludes by summarizing the original results presented in the dissertation as well as future projects we will be pursuing based on this work.

Chapter 2

2-to-2 Scattering and Helicity Eigenstates

2.1 Chapter Summary

This chapter establishes various definitions and conventions from four-dimensional (4D) quantum field theory which are relevant to this dissertation, e.g. that we use the 'mostly-minus' Minkowski metric and all indices are raised/lowered with the Minkowski metric. It is written with the aim of providing a self-consistent collection of standard derivations which all use the same conventions. This is done under the belief that such a collection could be useful to other physicists. As such, many details and observations are intentionally included which are often skipped in standard resources. For physicists who are already familiar with 2-to-2 scattering calculations involving helicity eigenstates, much of this chapter can be skimmed without missing details vital to the remainder of this dissertation.

This chapter is organized as follows:

- Section 2.2 derives the Lorentz and Poincaré groups from the assumption that the speed of light is globally invariant between reference frames. Active forms for the Poincaré transformations (rotations, boosts, spacetime translations) and their generators ($\vec{J}$, $\vec{K}$, P^{μ}) are provided in the 4-vector representation, and the commutation structure of the generators is derived. The section closes by deriving the Lorentz-invariant phase space.

- Section 2.3 promotes the Poincaré generators to Hermitian operators and thus promotes the corresponding transformations to unitary operators. The helicity operators is introduced.

- Section 2.4 defines single-particle 4-momentum external states, which are then combined to form multiparticle 4-momentum external states. Special care is taken to consider multiparticle states involving identical particles. The S-matrix element is introduced and its relation to the matrix element $\mathcal{M}$ is mentioned.

- Section 2.5 describes 2-to-2 particle processes in detail, with emphasis on scattering in the center-of-momentum (COM) frame and parameterization via the Mandelstam variables. An equation for simplifying integrals over the 4-momenta of two particles is derived and then applied to unitarity of the S-matrix in order to derive the optical theorem.

- Section 2.6 summarizes the usual treatment of angular momentum in quantum mechanics, including how angular momentum representations are combined, and defines the Wigner-D matrix.
- Section 2.7 considers single-particle helicity eigenstates, which are combined to form multiparticle helicity eigenstates. Using the relationship between helicity eigenstates and angular momentum eigenstates, the matrix element is decomposed in an angular momentum basis in order to define partial wave amplitudes. The elastic and inelastic partial wave unitarity constraints are derived.
- Section 2.8 derives the spin-1 and spin-2 polarization structures. Various canonical quadratic Lagrangians are considered, and their corresponding propagators are listed.

2.2 Poincaré Group: 4-Vector Representation

2.2.1 Preserving the Speed of Light

At the heart of modern relativity theory lies an axiom with far-reaching consequences: no matter how different the reference frames of two observers, they will agree that a wavepacket of light travels at a speed c. This defines the aptly-named speed of light.

Every reference frame is characterized by a choice of coordinates, which presently means a unique continuous association of every point of reality with a time coordinate $ct = x^0$ and some spatial coordinates $\vec{x} = (x^1, x^2, x^3)$. In such a reference frame, a wavepacket of light will travel along some curve $\vec{x}(t)$ through three-dimensional space and, according to the aforementioned axiom of relativity, do so at the speed of light, such that $c = |d\vec{x}/dt|$; however, it is worthwhile to recast this universal property as an equation relating differentials along the motion of the wavepacket:

$$c = \left|\frac{d\vec{x}}{dt}\right| \qquad \Longrightarrow \qquad c|dt| = |d\vec{x}| \qquad \Longrightarrow \qquad c^2|dt|^2 - |d\vec{x}|^2 = 0 \tag{2.1}$$

This latter form is useful because it treats the space and time coordinates equivalently, with the speed of light amounting to a conversion from time duration units to length units. According to relativity theory, although an observer in a different inertial reference frame with different coordinates $(ct', \vec{x}\,')$ will measure that same wavepacket as traveling along a different trajectory $\vec{x}\,'(t')$, they will still find that its speed $|d\vec{x}\,'/dt|$ equals c at every point along its path, or equivalently

$$c^2|dt'|^2 - |d\vec{x}\,'|^2 = 0 \tag{2.2}$$

This invariance greatly restricts the structure of reality. Imagine flooding reality with wavepackets of light that propagate in all directions and at every point of time and space. By the axiom of relativity, an observer in any other reference frame must also agree that every wavepacket in this vast network travels at the speed of light, even if their own perception of spacetime is wildly different. This puts a tight constraint on the local structure of reality itself, and requires that space and time must be woven together into a unified manifold of spacetime.

Consider the possible 4-velocities $v^\mu \equiv (v^0, \vec{v}) = (v^0, v^1, v^2, v^3)$ of a trajectory passing through a certain spacetime point. If the trajectory describes the motion of a wavepacket of light according to Eq. (2.2), then the 4-velocity is light-like: $v^2 \equiv v \cdot v = 0$, where

$$(v \cdot v) \equiv \eta_{\mu\nu} v^\mu v^\nu \equiv \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} v^\mu v^\nu \tag{2.3}$$

and $\eta_{\mu\nu}$ is the Minkowski metric

$$[\eta_{\mu\nu}] = \text{Diag}(+1, -1, -1, -1) \equiv \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.4}$$

when expressed as a matrix with those components; this square bracket notation will be used throughout this dissertation. The metric is symmetric by construction ($\eta_{\mu\nu} = \eta_{\nu\mu}$) and we define it in the "mostly-minus" convention, i.e. it has one $+1$ eigenvalue and three -1 eigenvalues corresponding to temporal and spatial information respectively. We adopt the Einstein summation convention throughout the remainder of this dissertation, so that repeated indices indicate sums over the corresponding index ranges. Whenever we raise or lower a four-dimensional index, it is done with the Minkowski metric. In the next chapter where five-dimensional spacetimes are relevant, we raise and lower each five-dimensional index with the flat five-dimensional metric $[\eta_{MN}] \equiv \text{Diag}(+1, -1, -1, -1, -1)$.

For the sake of performing calculations, it is vital to generalize the above language to include generic 4-vectors, e.g. objects of the form $a = (a^0, a^1, a^2, a^3)$ for which $(a \cdot a)$ does not necessarily vanish. Through the Minkowski metric η, a generic 4-vector a^μ implies a related 4-covector a_μ

$$a_\mu = (a_0, a_1, a_2, a_3) \equiv \eta_{\mu\nu} a^\nu = (a^0, -a^1, -a^2, -a^3) \tag{2.5}$$

and for generic 4-vectors a and b the previous inner product generalizes to

$$(a \cdot b) = \eta_{\mu\nu} a^\mu b^\nu = a_\mu b^\mu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 \tag{2.6}$$

where $(a \cdot a)$ is called the magnitude of a. Sometimes we will break a 4-vector a into its temporal a^0 and spatial a^i components, the latter of which comprise a 3-vector $\vec{a} = (a^1, a^2, a^3)$. 3-vectors are defined with the usual 3-vector inner product, i.e.

$$\vec{a} \cdot \vec{b} = a^i b^i = a^1 b^1 + a^2 b^2 + a^3 b^3 \tag{2.7}$$

such that the 4-vector inner product equals

$$a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b} \tag{2.8}$$

To avoid confusion, four-dimensional (4D) spacetime indices are labeled via lowercase Greek letters (μ, ν, ρ, ...) with $\mu \in \{0, 1, 2, 3\}$, whereas three-dimensional (3D) spatial indices are labeled via lowercase Latin letters (i, j, k, ...) with $i \in \{1, 2, 3\}$. In the next chapter, we consider five-dimensional (5D) spacetime indices, which are labeled via uppercase Latin letters (M, N, R, ...) with $M \in \{0, 1, 2, 3, 5\}$. The 3-vector components will sometimes be relabeled to make contact with the usual (x, y, z)-rectilinear 3-space coordinates, in which case $a_x \equiv a^1$, $a_y \equiv a^2$, and $a_z \equiv a^3$.

Returning to the invariance of the speed of light, consider the classification of all invertible linear transformations λ that preserve light-like magnitudes:

$$v \cdot v = \eta_{\mu\nu} v^\mu v^\nu = 0 \qquad \Longrightarrow \qquad (\lambda v) \cdot (\lambda v) = \eta_{\mu\nu} (\lambda v)^\mu (\lambda v)^\nu = 0 \tag{2.9}$$

As previously mentioned, demanding invariance of this inner product for all light-like 4-vectors is a significant constraint. By expressing a generic 4-vector as a sum of light-like 4-vectors, it can be demonstrated that preserving light-like inner products necessarily implies the preservation of *all* inner products between 4-vectors up to a net rescaling. That is,

$$(\lambda a) \cdot (\lambda b) = \Omega\, (a \cdot b) \tag{2.10}$$

for a positive real number Ω. Therefore, the linear transformation λ decomposes into the composition of a dilation by an amount $\sqrt{\Omega}$ and a Lorentz transformation Λ like so:

$$\lambda = \sqrt{\Omega}\Lambda \tag{2.11}$$

where $|\det(\Lambda)| = 1$ characterizes the Lorentz transformation. The dilation simply scales our time duration and length units by an equal amount $\sqrt{\Omega}$. Because we are interested in comparing reference frames that differ beyond a choice of units, we set $\Omega = 1$ so that $\lambda = \Lambda$, and we from here on restrict our attention to Lorentz transformations.

Lorentz transformations preserve 4-vector magnitude, and therefore magnitudes can be classified in a frame-independent way: given a 4-vector a, it is said to be space-like, light-like, or time-like if its magnitude is less than, equal to, or greater than 0 respectively. These names are inspired by considering a spacetime displacement ℓ^μ from the origin. If its magnitude vanishes $(\ell \cdot \ell) = 0$, then it is a displacement that could be traversed by a wavepacket of light. Meanwhile, a pure spatial displacement $\ell = (0, \vec{\ell})$ yields a negative magnitude $(\ell \cdot \ell) = -\vec{\ell} \cdot \vec{\ell} < 0$, and a pure temporal displacement $\ell = (\ell^0, \vec{0})$ yields a positive magnitude $(\ell \cdot \ell) = (\ell^0)^2 > 0$, and thus they are space-like and time-like respectively. A time-like (space-like) particle velocity corresponds to motion slower (faster) than the speed-of-light, and a trajectory is labeled space-, light-, or time-like if every 4-velocity along that trajectory is also space-, light-, or time-like respectively. Similarly, a 3-dimensional hypersurface in 4D spacetime (or, more generally, a $(X-1)$-dimensional hypersurface in X-dimensional spacetime) is labeled space-, light-, or time-like if every normal to that hypersurface is time-, light-, or space-like respectively (be careful to note that this latter ordering of descriptors is reversed relative to the others).

A Lorentz 4-vector is any 4-vector (4-velocity or otherwise) that transforms under a Lorentz transformation in the way previously described: that is, the Lorentz 4-vector v^μ goes to another Lorentz 4-vector $\overline{v}^\mu$ after a Lorentz transformation Λ, where

$$\overline{v}^\mu \equiv \Lambda^\mu{}_\nu v^\nu \tag{2.12}$$

An index such as ν in v^ν which is transformed by contraction with $\Lambda^\mu{}_\nu$ under a Lorentz transformation Λ is called a contravariant index. Because $(\Lambda a)\cdot(\Lambda b) = (a\cdot b)$ for all 4-vectors a and b, Lorentz transformations preserve the metric in the following sense,

$$\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma} = \eta_{\mu\nu} \tag{2.13}$$

Furthermore, the Lorentz transformations define a group under composition (i.e. $(\Lambda_1)^\mu{}_\nu (\Lambda_2)^\nu{}_\rho = (\Lambda_3)^\mu{}_\rho$), with a transformation Λ related to its inverse Λ^{-1} according to

$$(\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu \tag{2.14}$$

because

$$\Lambda_\nu{}^\mu \Lambda^\nu{}_\rho = [\eta^{\mu\tau}\eta_{\sigma\nu}\Lambda^\sigma{}_\tau]\,\Lambda^\nu{}_\rho = \eta^{\mu\tau}\left[\Lambda^\sigma{}_\tau \Lambda^\nu{}_\rho \eta_{\sigma\nu}\right] = \eta^{\mu\tau}\eta_{\tau\rho} = \eta^\mu_\rho \tag{2.15}$$

and $[\eta^\mu_\rho] = \mathrm{Diag}(+1,+1,+1,+1) = \mathbb{1}$. Thus, we refer to the collection of all Lorentz transformations as the Lorentz group.

The Lorentz group can be further divided into four distinct connected components based on the determinant and temporal-temporal component of each transformation Λ:

- If $\det\Lambda = +1$ then Λ is proper. Otherwise, $\det\Lambda = -1$ and Λ is improper.
- If $\Lambda_{00} \geq 1$, then Λ is orthochronous. Otherwise, $\Lambda_{00} \leq -1$, and Λ is antichronous.

These different connected components can be mapped onto one-another via the discrete Lorentz transformations P and T,

$$[P^\mu{}_\nu] = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad [T^\mu{}_\nu] = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} \tag{2.16}$$

and their combined action $PT = TP$, where P and T are called the parity-inversion and time-reversal transformations respectively. We are most concerned with proper orthochronous Lorentz transformations, which are continuously connected to the identity transformation and form a subgroup of the wider Lorentz group. In fact, we will use this group so often that we drop the "proper orthochronous" descriptor from hereon: unless otherwise indicated, these are the transformations to which we refer when discussing the Lorentz group.

The transformation behavior of a Lorentz 4-vector can be used to derive the transformation behaviors of other Lorentz tensors. For example, a Lorentz 4-covector v_μ becomes another Lorentz 4-covector $\overline{v}_\mu$ under the Lorentz transformation Λ according to

$$v_\mu = \eta_{\mu\nu} v^\nu \quad \mapsto \quad \overline{v}_\mu = \eta_{\mu\nu}\overline{v}^\nu = \eta_{\mu\nu}\Lambda^\nu{}_\rho v^\rho = \Lambda_\mu{}^\rho v_\rho = (\Lambda^{-1})^\rho{}_\mu v_\rho \tag{2.17}$$

As illustrated by the above result, symbols that require both the inversion label "-1" and Lorentz indices are cumbersome. This is not the only time the inversion label clutters otherwise convenient notations that are useful to this dissertation, so we will instead write inverses with a tilde, e.g. $\tilde{\Lambda}^\mu{}_\nu \equiv (\Lambda^{-1})^\mu{}_\nu$. In this notation, the transformed 4-covector is more succinctly written as $\overline{v}_\mu = \tilde{\Lambda}^\rho{}_\mu v_\rho$. A Lorentz index ν that transforms via contraction with $\tilde{\Lambda}^\mu{}_\nu$ is called a covariant index.

More generally, a Lorentz tensor $X^{\alpha_1\cdots\alpha_a}{}_{\beta_1\cdots\beta_b}$ is an object with a contravariant indices α_1, ..., α_a and b covariant indices β_1, ..., β_b that transforms under a Lorentz transformation Λ according to

$$X^{\alpha_1\cdots\alpha_a}{}_{\beta_1\cdots\beta_b} \mapsto \Lambda^{\alpha_1}{}_{\gamma_1}\cdots\Lambda^{\alpha_a}{}_{\gamma_a}\tilde{\Lambda}^{\delta_1}{}_{\beta_1}\cdots\tilde{\Lambda}^{\delta_b}{}_{\beta_b}X^{\gamma_1\cdots\gamma_a}{}_{\delta_1\cdots\delta_b} \tag{2.18}$$

A tensor that transforms according to this rule is said to transform covariantly under Lorentz transformations or, in fewer words, to be Lorentz covariant. By contracting Lorentz indices between Lorentz tensors, a new Lorentz tensor can be formed. In particular, if all Lorentz indices are contracted within a product of Lorentz tensors (and the collection possesses no other transformation properties with regards to Lorentz transformations) then a Lorentz scalar is formed. For example, the inner product $(v\cdot v) = v^\mu v_\mu$ is a Lorentz scalar, and is thereby invariant under Lorentz transformations. Our field theory Lagrangian densities will also be Lorentz scalars. In particular, the Lagrangians we consider will be constructed from multiple rank-2 tensors $\hat{h}^{(n)}_{\mu\nu}$ corresponding to spin-2 fields. These nicely contract like links in a chain, and their contractions are so common that it is worthwhile to grant them a special notation. We define the 'twice-squared bracket' notation as follows:

$$[\![\hat{h}^{(n_1)}]\!]_{\mu\nu} \equiv \hat{h}^{(n_1)}_{\mu\nu} \tag{2.19}$$

$$[\![\hat{h}^{(n_1)}\hat{h}^{(n_2)}]\!]_{\mu\sigma} \equiv \hat{h}^{(n_1)}_{\mu\nu}\eta^{\nu\rho}\hat{h}^{(n_2)}_{\rho\sigma} \tag{2.20}$$

$$[\![\hat{h}^{(n_1)}\hat{h}^{(n_2)}\hat{h}^{(n_3)}]\!]_{\mu\upsilon} \equiv \hat{h}^{(n_1)}_{\mu\nu}\eta^{\nu\rho}\hat{h}^{(n_2)}_{\rho\sigma}\eta^{\sigma\tau}\hat{h}^{(n_2)}_{\tau\upsilon} \tag{2.21}$$

and so on. When the field indices are entirely contracted to form a trace (such that the chain is closed into a loop), the external indices are omitted:

$$[\![\hat{h}^{(n_1)}\cdots\hat{h}^{(n_H)}]\!] \equiv [\![\hat{h}^{(n_1)}\cdots\hat{h}^{(n_H)}]\!]_{\alpha\beta}\eta^{\alpha\beta} \tag{2.22}$$

The operation of connecting two such chains via contraction is called concatenation, and the identity chain with respect to concatenation is

$$[\![1]\!]_{\mu\nu} \equiv \eta_{\mu\nu} \tag{2.23}$$

from which $[\![1]\!] = 4$. (If we were instead working in X-dimensions, then $[\![1]\!]_{MN} \equiv \eta_{MN}$ and $[\![1]\!] = X$).

Regarding its group structure, the Lorentz group possesses two Casimir invariants, which are used to define particle content in quantum field theory. The first is the mass, which is defined from the (assumedly not space-like) 4-momentum $p^\mu = (E/c, \vec{p})$ where $E \geq 0$ and $\vec{p}$ are the energy and 3-momentum of a particle excitation respectively. The mass $m \geq 0$ is defined from the Einstein equation,

$$E^2 = m^2c^4 + \vec{p}^2c^2 \tag{2.24}$$

which we typically express instead as the on-shell condition $p^2 \equiv p^\mu p_\mu = m^2c^2$. The collection of light-like 4-momenta related by Lorentz transformations form the light cone, a right cone in p^μ-space oriented along the energy axis. In contrast, if a 4-momentum is time-like, then the mass is nonzero, and the collection of 4-momenta with equal mass form a hyperboloid in p^μ-space called a mass hyperboloid. Every mass hyperboloid contains a rest frame 4-momentum $(m, \vec{0})$ wherein $|\vec{p}| = 0$. Any two 4-momenta on the light-cone or on the same mass hyperboloid can be related via a Lorentz transformation. The mass additionally dictates the kind of trajectories along which a given particle can travel: massless particles travel along light-like trajectories at the speed of light, whereas massive particles travel along time-like trajectories at speeds slower than the speed of light.

Regarding the second Casimir (the Pauli-Lubanski pseudovector), we will not dwell on it beyond asserting that it allows a massive (massless) particle to be assigned a Lorentz-invariant total spin (helicity). For instance, the second Casimir invariant is why an electron can be assigned a definite internal spin of $\frac{1}{2}$. We adopt the standard convention of referring to a massless particle with total helicity s as being a spin-s particle. When a massive particle is in its rest frame, its total angular momentum equals its total internal spin.

The above considerations for 4-momentum apply more generally to other Lorentz 4-vectors as well: any two (nonzero) light-like or time-like 4-vectors v and w having equal magnitude $(v \cdot v) = (w \cdot w) \geq 0$ and same temporal component sign $\text{sign}(w^0) = \text{sign}(v^0)$ can be related by a Lorentz transformation. Meanwhile, any two space-like 4-vectors v and w with equal magnitude $(v \cdot v) = (w \cdot w) < 0$ can be related by a Lorentz transformation, regardless if they disagree on the signs of their temporal components. The collection of all 4-vectors related to a 4-vector v by (proper orthochronous) Lorentz transformations is called the Lorentz-invariant hypersurface generated by v. In this language, a mass hyperboloid (light cone) is the Lorentz-invariant hypersurface generated by a time-like (light-like) 4-momentum p. Note that the Lorentz-invariant hypersurface generated by a nonzero 4-vector is a three-dimensional manifold because the four components of the 4-vectors on that hypersurface have only one continuous constraint (i.e. maintaining the same overall 4-vector magnitude). The "nonzero" descriptor in the previous statement is important because the 4-vector origin 0^μ is individually invariant under Lorentz transformations, such that the hypersurface it generates is the zero-dimensional set $\{0^\mu\}$.This is one way to understand the lack of a rest frame 4-momentum on the light cone: if we could somehow map the light-like 4-momentum of a massless particle to 0^μ, then we could (using the inverse transformation) map 0^μ back to a different light-like 4-momentum, but this would contradict the invariance of $\{0^\mu\}$. Therefore, massless particles cannot be at rest in any reference frame (this is, of course, a restatement of the invariance of the speed of light).

In addition to its group structure, the Lorentz group forms a six-dimensional manifold: consider the magnitude

$$a \cdot a = (a^0)^2 - (a^1)^2 - (a^2)^2 - (a^3)^2 \tag{2.25}$$

of a 4-vector a^μ for which all components are nonzero. A generic Lorentz transformation can alter any of these components but must ultimately preserve this magnitude. In particular, suppose a transformation alters one component slightly. Because all components of a^μ are assumedly nonzero, we can preserve the overall magnitude of a by slightly increasing or decreasing a different component of a by however much is necessary to accommodate the change of the first component. There are as many independent ways of performing this balancing trick as there are distinct pairs of components. Because a has four components as a 4-vector, there are six independent choices of component pairs. Furthermore, by chaining together the shifts of magnitude described by these six independent component pairs, we can form any (proper orthochronous) Lorentz transformation. Therefore, the Lorentz group is six-dimensional.

It is conventional to distinguish certain convenient Lorentz transformations:

- Rotations are Lorentz transformations that leave the temporal coordinate unchanged, and correspond to the usual collection of rotations in 3-space. Their operation solely affects the 3-vector part $\vec{a}$ of a 4-vector a, and they form a closed subgroup of the Lorentz group. In the context of the aforementioned balancing trick, these transformations correspond to the "space-space" mixing.

- Boosts are Lorentz transformations that leave a spatial 2-plane unchanged, e.g. a boost along the z-axis will mix the a^0 and a^3 components of a 4-vector, but leave the a^1 and a^2 components unchanged. Boosts do not form a closed subgroup of the Lorentz group. In the context of the aforementioned balancing trick, these transformations correspond to the "time-space" mixing.

Any two 4-vectors on the same Lorentz-invariant hypersurface can be related by a Lorentz transformation that combines rotations and boosts.

We arrived at the Minkowski metric by demanding that the speed of light be locally preserved between frames. If we now suppose the Minkowski metric describes spacetime globally as well (thereby ensuring we work in the realm of special relativity as opposed to general relativity), the trajectories $\vec{x}(t)$ of light-like wavepackets must be straight lines through 3-space. That is, the wavepacket propagates such that at any time t it is centered at $\vec{x}(t) = \vec{v}t + \vec{x}(0)$ for some initial position $\vec{x}(0)$ and velocity $|\vec{v}| = c$. If the 4-velocity $(v^0, \vec{v})$ transforms according to a Lorentz transformation

$$v^\mu \to \Lambda^\mu{}_\nu v^\nu \tag{2.26}$$

then the corresponding trajectory in 4-space $x^\mu = (ct, \vec{x}(t))$ must transform according to the same Lorentz transformation plus a potential spacetime translation

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + \epsilon^\mu \tag{2.27}$$

where ϵ^μ is a generic 4-vector. By once again considering a network of light-like wavepackets throughout spacetime, we can generalized this transformation behavior beyond a single trajectory and conclude that the coordinates of spacetime must generally transform according to Eq. (2.27). The wider collection of transformations available to spacetime coordinates comprise the Poincaré group. Because ϵ^μ has four real components and the Lorentz group is a six-dimensional manifold, the Poincaré group forms a ten-dimensional manifold.

The following subsections delve into more detail about specific transformations within the Poincaré group. To facilitate succinct expressions, we introduce unit 4-vector basis elements,

$$v = v^0\hat{t} + v^1\hat{x} + v^2\hat{y} + v^3\hat{z} \tag{2.28}$$

where

$$[\hat{t}^\mu] = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad [\hat{x}^\mu] = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad [\hat{y}^\mu] = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad [\hat{z}^\mu] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.29}$$

For the same purpose, we also define abbreviations for the trigonometric and hyperbolic functions

$$c_\alpha \equiv \cos\alpha \qquad s_\alpha \equiv \sin\alpha \qquad ch_\beta \equiv \cosh\beta \qquad sh_\beta \equiv \sinh\beta \tag{2.30}$$

and utilize natural units for the remainder of this dissertation: $c = \hbar = 1$.

2.2.2 Active vs. Passive Transformations

In order to quantify Lorentz and Poincaré transformations, we must decide whether to consider them as active or passive transformations. In order to clarify the nuances of these perspectives, let us briefly restrict our attention to spacetime translations.

Consider a continuous function $\phi(x)$ of real numbers over spacetime that is sharply peaked at some spacetime point $x = X$ relative to an observer at the spacetime origin. Further suppose we want to describe this same distribution as instead having a peak at $X + a$ for some 4-vector a relative to that observer. We might use an active or passive transformation to achieve this: the active transformation shifts the entire distribution by an amount a relative to the coordinate system, whereas the passive transformation instead keeps the distribution as-is and moves the observer (and the spacetime origin with them) by an amount $-a$. Because they ultimately describe the same physical reality—namely, that the peak is now at $X + a$ relative to the observer—these different transformations must be physically equivalent. More generally, an active Poincaré transformation $\mathcal{P}(\Lambda, a)$ on the distribution corresponds to a passive Poincaré transformation $\mathcal{P}(\Lambda, a)^{-1}$ on the coordinates.

When switching between reference frames via a transformation, the passive interpretation is the more popular choice: in this interpretation, reality is fixed, and we are merely swapping between observers who have their own coordinate systems for observing that reality. However, the preceding discussion points out that we could equally well use active transformations as long as we are careful to invert the intended operation. Consequently, because we intend to eventually apply active transformations to quantum mechanical states, our discussion of the Lorentz group in the upcoming subsections is written in the active interpretation, even when those transformations are used to switch between reference frames. For example, our rotation operator $R_z(\alpha)$ corresponds to rotating *the physical system* by an angle $+\alpha$ about the z-axis, which is equivalent to rotating *the observer* (and their coordinate system) by an angle $-\alpha$ about the z-axis. These are an active and passive transformation respectively.

That being said, there is an important transformation that we should be cautious to ensure is always interpreted correctly: the time evolution transformation. An active time translation by an amount Δt shifts our distribution $\phi(x) = \phi(t, \vec{x})$ to $\phi(t - \Delta t, \vec{x})$ and thereby ensures that a peak formerly at $X = (T, \vec{X})$ will subsequently occur at $X' = (T + \Delta t, \vec{X})$.

However, if we want to evolve the system in time by an amount Δt, we actually desire that $\phi(x, \vec{x})$ be mapped to $\phi(t + \Delta t, \vec{x})$. This can be achieved by either performing an active time translation by an amount $-\Delta t$ or (as it is usually expressed) performing a passive time translation by an amount Δt.

From here onward, the "rotation", "boost", "translation", "Lorentz", "Poincaré", etc. transformations will be written as active transformations unless otherwise indicated, in contrast to the time evolution transformation, which (in the way just described) is always understood as a passive transformation.

2.2.3 Rotations

For spatial coordinates, we utilize a standard right-handed 3-space coordinate system (labeled such that $\hat{x} \times \hat{y} = \hat{z}$) and define our rotations using the right-hand rule. This means that, for example, an active rotation about the z-axis by an angle α on a generic 4-vector x^μ yields $R_z(\alpha)^\mu{}_\nu x^\nu$, where

$$[R_z(\alpha)^\mu{}_\nu] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.31}$$

within which $c_\alpha \equiv \cos\alpha$ and $s_\alpha \equiv \sin\alpha$. When $\alpha = 0$, $R_z(\alpha)$ becomes the identity transformation. That this is a Lorentz transformation can be checked directly from the defining property of a Lorentz transformation, Eq. (2.13):

$$\begin{aligned} \eta_{\mu\nu}\ [R_z(\alpha)x]^\mu\ [R_z(\alpha)x]^\mu &= dt^2 - (c_\alpha\, dx - s_\alpha\, dy)^2 - (s_\alpha\, dx + c_\alpha\, dy)^2 - dz^2 && (2.32) \\ &= dt^2 - (c_\alpha^2 + s_\alpha^2)dx^2 - (c_\alpha^2 + s_\alpha^2)dy^2 - dz^2 && (2.33) \\ &= dt^2 - d\vec{x}^{\,2} && (2.34) \\ &= \eta_{\mu\nu}x^\mu x^\nu && (2.35) \end{aligned}$$

In principle, $R_z(\alpha)$ is an instantaneous mapping from one coordinate system to another. However, by taking $\alpha \to 0$, $R_z(\alpha)$ continuously goes to the identity ($[R_z(\alpha)^\mu{}_\nu] \to [\eta^\mu_\nu] = [\delta_{\mu,\nu}]$), and thus (by reversing the direction of the limit) we can interpret a rotation $R_z(\alpha)$ as a continuous transformation that smoothly rotates x^μ to $R_z(\alpha)^\mu{}_\nu x^\nu$. In addition to being a nice conceptual feature, this continuity near the identity allows us to rewrite the rotation operator $R_z(\alpha)$ as the exponential of a generator J_z:

$$[R_z(\alpha)^\mu{}_\nu] = \text{Exp}\Big[\alpha[(J_z)^\mu{}_\nu]\Big] \equiv \sum_{n=0}^{+\infty} \frac{1}{n!}\Big(\alpha[(J_z)^\mu{}_\nu]\Big)^n \qquad \text{where} \qquad J_z \equiv \left.\frac{\partial R_z(\alpha)}{\partial\alpha}\right|_{\alpha=0} \tag{2.36}$$

from which we calculate

$$[(J_z)^\mu{}_\nu] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.37}$$

By having one index raised and another index lowered, we ensure that powers of $[(J_z)^\mu{}_\nu]$ correctly reproduce a series of index contractions, e.g. $[(J_z)^\mu{}_\nu]\,[(J_z)^\nu{}_\rho] = [(J_z)^\mu{}_\rho]$. For the rest of this chapter we will drop the index references on $[(J_z)^\mu{}_\nu]$ and refer to it simply as J_z. Note that J_z only leaves 4-vectors proportional to $(t, \hat{z})$ unchanged, which is consistent with $\hat{z}$ being the axis of the rotation generated by J_z. This same procedure can also be applied to rotations about the x- and y-axes, which have the rotation matrices,

$$[R_x(\alpha)^\mu{}_\nu] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_\alpha & -s_\alpha \\ 0 & 0 & s_\alpha & c_\alpha \end{pmatrix} \qquad [R_y(\alpha)^\mu{}_\nu] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & 0 & s_\alpha \\ 0 & 0 & 1 & 0 \\ 0 & -s_\alpha & 0 & c_\alpha \end{pmatrix} \tag{2.38}$$

which can be expressed as $R_x(\alpha) = \text{Exp}[\alpha J_x]$ and $R_y(\alpha) = \text{Exp}[\alpha J_y]$, where

$$J_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{pmatrix} \qquad J_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{2.39}$$

(When we promote these to the quantum description, these antisymmetric matrices will be replaced by Hermitian operators.) The generators J_i have several convenient properties. For instance, they possess a closed commutator structure:

$$[J_i, J_j] = \epsilon_{ijk} J_k \qquad \Longrightarrow \qquad \vec{J} \times \vec{J} = \vec{J} \tag{2.40}$$

where $i, j, k \in \{x, y, z\}$, $\vec{J} \equiv (J_x, J_y, J_z)$, and $[A, B] \equiv AB - BA$. They can also be put into the combination

$$\vec{J}^2 \equiv \vec{J} \cdot \vec{J} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \equiv -2(\delta_{\mu,\nu} - \delta_{\mu,0}\delta_{\nu,0}) \tag{2.41}$$

which commutes with every generator

$$[\vec{J}, \vec{J}^2] = 0 \tag{2.42}$$

If a collection of three tensors $\{X_x, X_y, X_z\}$ happen to satisfy

$$[J_i, X_j] = \epsilon_{ijk} X_k \tag{2.43}$$

where $i, j, k \in \{x, y, z\}$, then the collection transforms like a 3-vector $\vec{X} \equiv (X_x, X_y, X_z) \equiv (X^1, X^2, X^3)$ under rotations. In particular, via Eq. (2.40), $\vec{J}$ transforms as a proper 3-vector under rotations, and so we can give the components of $\vec{J}$ legitimate 3-vector indices: $\{J_x, J_y, J_z\} = \{J^1, J^2, J^3\}$. Because we use the mostly-minus metric convention, this means that, for example, $J_x = J^1 = -J_1$. Exponentiating the generators together allows us to write a generic rotation matrix: a rotation $[R(\vec{\alpha})^\mu_\nu]$ around an axis $\hat{\alpha}$ by an angle $|\vec{\alpha}|$ equals

$$R(\vec{\alpha}) \equiv \text{Exp}[\vec{\alpha} \cdot \vec{J}] \tag{2.44}$$

This is, of course, equivalent to a rotation by an angle $-|\vec{\alpha}|$ about $-\hat{\alpha}$ instead, if one so prefers.

Note that the rotation generator set $\{J_x, J_y, J_z\}$ is closed under the commutation bracket. In fact, the generators $\{J_x, J_y, J_z\}$ form the Lie algebra $\mathfrak{so}(3)$ and the rotation group in three dimensions forms the Lie group $SO(3)$. The operator $\vec{J}^2$ (which we recall commutes with every generator) is the single Casimir operator belonging to this Lie algebra. Like other Casimir operators, $\vec{J}^2$ is a geometric invariant that describes the dimensionalities of any invariant subgroups within a given representation of the rotation group. For example, although the 4-vector representation above transforms under the rotation group in a well-defined way, it actually contains two distinct rotational behaviors which never mix under any rotation, as hinted by the two distinct eigenvalues along the diagonal of $\vec{J}^2$ in Eq. (2.41). This can also be identified directly from the transformation behavior of 4-vectors if one knows what to search for: while the 3-vector part $\vec{x}$ of a 4-vector x^μ is changed under the rotation in the usual way, its temporal component x^0 is left invariant, and so x^μ cleanly separates into x^0 and $\vec{x}$ as far as rotations are concerned. Regardless of how these invariant subspaces are derived, they correspond to spin-0 and spin-1 representations of the rotation group. We use the spin-1 portion of the 4-vector representation to derive the canonical spin-1 and spin-2 polarizations in Subsection 2.8.1, after we have promoted the Poincaré generators to quantum operators.

Eq. (2.44) is only one of many ways of writing a generic rotation. Another (which is particularly useful for the purposes of this chapter) is the Euler angle parameterization. The Euler angles detail a sequence of rotations with which one can produce any orientation of a rigid body in 3-space. They also happen to be a natural coordinate system for a symmetric top. Explicitly, we may write a generic rotation in terms of the Euler angles $\{\phi, \theta, \psi\}$ as

$$R(\phi, \theta, \psi) \equiv R_z(\phi) R_y(\theta) R_z(\psi) \tag{2.45}$$

where $\phi \in [0, 2\pi)$, $\theta \in [0, \pi]$, and $\psi \in (-2\pi, 0]$. When applied to a symmetric top which has been set to balance with its tip at the origin and with gravity pulling in the negative $\hat{z}$-direction, these angles correspond to the following motions:

- ψ describes the intrinsic rotation of the top about its own axis.
- θ describes nutation of the top, i.e. rotation of the top axis towards and away from the z-axis.
- ϕ describes precision of the top, i.e. rotation of the top axis about the z-axis.

In quantum mechanical problems where the relevant states are eigenkets of z-axis rotations but (necessarily) not of x- and y-axis rotations, the fact that Eq. (2.45) begins and ends with z-axis rotations enables certain simplifications.

If the object we intend to rotate has no spatial extent beyond its axis of rotation (e.g. a symmetric top in the limit that it becomes a needle), then the intrinsic rotation angle ψ has no physical effect and can be set to some conventional value. This will be relevant when we consider rotations of 3-momenta, which can be rotated about their 3-direction without affecting their value. Popular conventions include setting $\psi = -\phi$ and $\psi = 0$, of which we choose the former when such a choice is relevant. Setting the value of ψ ensures that only two degrees of freedom remain, which is consistent with the spherical coordinate angles (θ, ϕ). For these cases, we define

$$R(\hat{p}) \equiv R(\phi, \theta) \equiv R(\phi, \theta, -\phi) \tag{2.46}$$

where $\hat{p}$ is the 3-direction corresponding to (θ, ϕ).

To phrase the previous point in a different way: any two 3-vectors $\vec{v}$ and $\vec{w}$ which share the same magnitude $|\vec{v}| = |\vec{w}|$ are on the same rotation invariant hypersurface, and can be related via some choice of rotation. Because these hypersurfaces are 2-spheres in 3-space, we require only two degrees of freedom to parameterize the different 3-vectors and, thus, the rotations relating them too. This is the language we use when discussing Lorentz transformations in the next subsection, after we derive the boost generators.

2.2.4 Boosts

An active boost along the z-axis with rapidity β on a generic 4-vector x^μ yields $B_z(\beta)^\mu{}_\nu x^\nu$, where

$$[B_z(\beta)^\mu{}_\nu] = \begin{pmatrix} ch_\beta & 0 & 0 & sh_\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ sh_\beta & 0 & 0 & ch_\beta \end{pmatrix} \tag{2.47}$$

within which $ch_\beta \equiv \cosh\beta$ and $sh_\beta \equiv \sinh\beta$. When $\beta = 0$, $B_z(\beta)$ becomes the identity transformation. Like the rotations in the last subsection, the fact that $B_z(\beta)$ is a Lorentz transformation can be checked directly from their defining property Eq. (2.13):

$$\eta_{\mu\nu}\ [B_z(\beta)x]^\mu\ [B_z(\beta)x]^\mu = (ch_\beta\, dt + sh_\beta\, dz)^2 - dx^2 - dy^2 - (sh_\beta\, dt + ch_\beta\, dz)^2 \tag{2.48}$$

$$= (ch_\beta^2 - sh_\beta^2)dt^2 - dx^2 - dy^2 - (ch_\beta^2 - sh_\beta^2)dz^2 \tag{2.49}$$

$$= dt^2 - d\vec{x}^{\,2} \tag{2.50}$$

$$= \eta_{\mu\nu} x^\mu x^\nu \tag{2.51}$$

Furthermore, because $B_z(\beta)$ is continuously connected to the identity, it can be interpreted as a smooth transformation (by evolving the rapidity from zero to β) and be expressed as an exponential of a generator:

$$[B_z(\beta)^\mu{}_\nu] = \text{Exp}\Big[\beta[(K_z)^\mu{}_\nu]\Big] \equiv \sum_{n=0}^{+\infty} \frac{1}{n!}\Big(\beta[(K_z)^\mu{}_\nu\Big)^n \qquad \text{where} \qquad K_z \equiv \frac{\partial B_z(\beta)}{\partial\beta}\bigg|_{\beta=0} \tag{2.52}$$

from which we calculate

$$[(K_z)^\mu{}_\nu] = \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix} \tag{2.53}$$

Again, we drop the index indicators and simply write $[(K_z)^{\mu}{}_{\nu}]$ as K_z. Boosts along the x- and y-axes are defined similarly

$$[B_x(\beta)^{\mu}{}_{\nu}] = \begin{pmatrix} ch_\beta & sh_\beta & 0 & 0 \\ sh_\beta & ch_\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad [B_y(\beta)^{\mu}{}_{\nu}] = \begin{pmatrix} ch_\beta & 0 & sh_\beta & 0 \\ 0 & 1 & 0 & 0 \\ sh_\beta & 0 & ch_\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.54}$$

and can be expressed as exponentials of generators $B_x(\beta) = \text{Exp}[\beta K_x]$ and $B_y(\beta) = \text{Exp}[\beta K_y]$, where

$$K_x = \begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_y = \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.55}$$

are the corresponding generators. Unlike the rotation generators, the boost generators are not closed with respect to the commutator bracket:

$$[K_i, K_j] = -\epsilon_{ijk} J_k \tag{2.56}$$

$$[J_i, K_j] = +\epsilon_{ijk} K_k \tag{2.57}$$

where $i, j, k \in \{x, y, z\}$. By comparing Eq. (2.57) to Eq. (2.43), we note $\{K_x, K_y, K_z\}$ do rotate like a proper 3-vector under rotations, and label them as such: $\vec{K} \equiv \{K_x, K_y, K_z\} = \{K^1, K^2, K^3\}$.

A generic boost $B(\vec{\beta})$ along an axis $\hat{\beta}$ by an amount β can be constructed from exponentiation of the generators:

$$B(\vec{\beta}) = \text{Exp}[\vec{\beta} \cdot \vec{K}] \tag{2.58}$$

which leaves the 2-plane perpendicular to $\vec{\beta}$ in 3-space unchanged.

The rotation and boost generators together form the set of Lorentz generators $\{\vec{L}, \vec{K}\}$, which enumerate six independent degrees of freedom and can generate any (proper orthochronous) Lorentz transformation via exponentiation. Like in the case of a generic rotation, there are many ways to parameterize a generic Lorentz transformation. One example (and the parameterization we choose) is as a boost followed by a rotation:

$$\Lambda(\vec{\alpha}, \vec{\beta})^{\mu}{}_{\rho} \equiv R(\vec{\alpha}\,)^{\mu}{}_{\nu}\, B(\vec{\beta}\,)^{\nu}{}_{\rho} \tag{2.59}$$

which has six degrees of freedom $(\vec{\alpha}, \vec{\beta})$. This number can be reduced if we only consider the Lorentz transformations that relate any two 4-vectors on the same Lorentz-invariant hypersurface. In particular, if the 3-vector part $\vec{v}$ of a 4-vector v points in a direction $\hat{v}$, then we can obtain any other 4-vector on the same Lorentz-invariant hypersurface by applying

$$\Lambda(\vec{\alpha}, \beta)^{\mu}{}_{\rho} \equiv R(\phi, \theta, -\phi)^{\mu}{}_{\nu}\, B(\beta\hat{v}\,)^{\nu}{}_{\rho} \tag{2.60}$$

for some choice of ϕ, θ, and β. Note that this specific collection of Lorentz transformations only has three degrees of freedom, consistent with the dimensionality of the light cone and mass hyperboloids.

In the next subsection, the translation generators are derived, which together with the Lorentz generators form the Poincaré generators.

2.2.5 Translations

Although it is unnecessary to do so in more general contexts, it is advantageous for our current purposes to cast the translation operation as a matrix. To do so, we extend 4-vectors for the duration of this subsection to include a new auxiliary slot, e.g.

$$x^\mu \sim [x^\mu] = \begin{pmatrix} x^\mu \\ 1 \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ 1 \end{pmatrix} \tag{2.61}$$

and define a translation operator $T(\epsilon)^\nu{}_\mu$ as a 5×5 matrix,

$$[T(\epsilon)^\nu{}_\mu] = \begin{pmatrix} \mathbb{1}^\nu_\mu & \epsilon^\nu \\ 0 & 1 \end{pmatrix} \tag{2.62}$$

where ϵ is a 4-vector, such that,

$$[T(\epsilon)^\nu{}_\mu x^\mu] = [T(\epsilon)^\nu{}_\mu]\,[x^\mu] = \begin{pmatrix} x^\nu + \epsilon^\nu \\ 1 \end{pmatrix} = [x^\nu + \epsilon^\nu] \tag{2.63}$$

Like the previous transformations, the translation operator can be generated through exponentiation of certain translation generators P^μ. However, let us be more careful about the signs in this exponentiation than we were in the rotation or boost cases. Specifically, to encourage Lorentz invariance, we would like to write the generators P^μ as a 4-vector contracted with a generating parameter ϵ^μ, so that the exponentiation is of the form

$$\mathrm{Exp}\left[\pm\left(\epsilon^0[(P^0)^\mu{}_\nu] - \epsilon^1[(P^1)^\mu{}_\nu] - \epsilon^2[(P^2)^\mu{}_\nu] - \epsilon^3[(P^3)^\mu{}_\nu]\right)\right] \tag{2.64}$$

where the sign of the exponent remains to be determined. The sign we choose is based on precedent: as written in Eqs. (2.44) and (2.58), the exponents of the equivalent expressions for general rotations and boosts equal $+\vec{\alpha}\cdot\vec{J}$ and $+\vec{\beta}\cdot\vec{K}$ respectively. It would be nice if the 3-vector part of the translation exponent equaled $+\vec{\epsilon}\cdot\vec{P}$ as well. Thus, we choose the lower sign.

Using this convention, the time-translation operator $H \equiv P^0$ is defined according to

$$[T(\epsilon^0\hat{t})^\nu{}_\mu] = \mathrm{Exp}\Big[-\epsilon^0[H^\mu{}_\nu]\Big] \qquad \text{where} \qquad H \equiv P^0 = \left.\frac{\partial T(\epsilon^0\hat{t})}{\partial \epsilon^0}\right|_{\epsilon^0=0} \tag{2.65}$$

from which we calculate

$$[H^\mu{}_\nu] = \begin{pmatrix} \mathbb{0}^\nu_\mu & -\hat{t}^\nu \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.66}$$

As before, we drop the index indicators on the generators as we proceed. Like the above temporal translation, a pure spatial translation

$$[T(\vec{\epsilon})^{\nu}{}_{\mu}] = \text{Exp}[\vec{\epsilon} \cdot \vec{P}] \qquad \text{where} \qquad P^i = \frac{\partial T(\vec{\epsilon})}{\partial \epsilon^i}\bigg|_{\epsilon^i=0} \tag{2.67}$$

is accomplished via the space-translation generators $\{P_x, P_y, P_z\}$, which explicitly equal

$$P_x \equiv P^1 = \begin{pmatrix} \mathbb{0}^{\nu}_{\mu} & \hat{x}^{\nu} \\ 0 & 0 \end{pmatrix} \qquad P_y \equiv P^2 = \begin{pmatrix} \mathbb{0}^{\nu}_{\mu} & \hat{y}^{\nu} \\ 0 & 0 \end{pmatrix} \qquad P_z \equiv P^3 = \begin{pmatrix} \mathbb{0}^{\nu}_{\mu} & \hat{z}^{\nu} \\ 0 & 0 \end{pmatrix} \tag{2.68}$$

Combining these yields a generic spacetime translation by an amount ϵ^{μ}:

$$[T(\epsilon^{\mu})] = \text{Exp}\left[-(\epsilon \cdot P)\right] \tag{2.69}$$

where $\hat{x}^{\mu}$, $\hat{y}^{\mu}$, and $\hat{z}^{\mu}$ were defined in Eq. (2.29). Every Poincaré transformation can be expressed as a combination of Lorentz transformations and spacetime translations, so we can now express all Poincaré transformations as products of exponentiations of generators.

Combining the spacetime translation generators with the Lorentz generators yields the ten canonical Poincaré generators $\{P^{\mu}, \vec{J}, \vec{K}\}$, where the Lorentz generators have implicitly been extended to accommodate the 5×5 forms of the translations, e.g. given a Lorentz generator G, a Poincaré generator $\overline{G}$ will have the same effect if defined as follows

$$\overline{G} = \begin{pmatrix} G & 0 \\ 0 & 1 \end{pmatrix} \tag{2.70}$$

We only distinguish the Poincaré generator $\overline{G}$ from the Lorentz generator G in the above definition. Otherwise, we just write G.

Via explicit evaluation, the commutator structure of the canonical Poincaré generators is found to be, in total,

$$[J^i, J^j] = +\epsilon_{ijk} J^k \qquad [J^i, K^j] = +\epsilon_{ijk} K^k \qquad [K^i, K^j] = -\epsilon_{ijk} J^k \tag{2.71}$$

$$[H, J^i] = 0 \qquad [H, K^i] = +P^i \qquad [J^i, P^j] = +\epsilon_{ijk} P^k \qquad [P^i, K^j] = +H\delta_{ij} \tag{2.72}$$

$$[P^{\mu}, P^{\nu}] = 0 \tag{2.73}$$

where $i, j, k \in \{1, 2, 3\}$. The commutators of the form $[J^i, \bullet]$ indicate that $\vec{P}$, $\vec{J}$, and $\vec{K}$ behave like 3-vectors under rotations, such that their 3-vector indices are meaningful.

2.2.6 Lorentz-Invariant Phase Space

Before promoting the generators to quantum operators in the next section, it is useful to derive a Lorentz-invariant integral measure with which we can eventually normalize our quantum states. Recall that a mass hyperboloid is a Lorentz-invariant hypersurface defined as the collection of 4-momentum p for which $E \equiv p^0 > 0$ and $(p \cdot p) > 0$. An integral over a given mass hyperboloid is easily expressed as a 4-momentum integral using these constraints

$$\int \frac{d^4p}{(2\pi)^4} \; (2\pi)\delta(p^2 - m^2) \; \theta(E) \; f(p) \tag{2.74}$$

where f is some function of the 4-momentum, the Dirac delta function $\delta(p^2 - m^2)$ enforces $p^2 = m^2$, and the Heaviside step function $\theta(E)$ enforces $E > 0$. Because the mass hyperboloid is a three-dimensional hypersurface, the goal of this subsection is to rewrite the 4-momentum integral Eq. (2.74) as a 3-momentum integral instead. First, we note that this 4-momentum integral is manifestly invariant under a Lorentz transformation Λ because

$$d^4p \mapsto d^4(\Lambda p) = |\det \Lambda| \, d^4p = d^4p \tag{2.75}$$

$$\delta(p^2 - m^2) \mapsto \delta\Big((\Lambda p)^2 - m^2\Big) = \delta(p^2 - m^2) \tag{2.76}$$

so long as $f(p)$ is a Lorentz scalar. We will first use the Dirac delta in order to eliminate the energy integral (dE in the decomposition $d^4p = dE \, d^3\vec{p}$). However, the Dirac delta as written is not quite right for eliminating that integral, because it is of the form

$$\delta(p^2 - m^2) = \delta(E^2 - \vec{p}^{\,2} - m^2) \tag{2.77}$$

instead of $\delta(E - E_*)$ for some value E_*. To get it into this form, we reparameterize the Dirac delta using the following property:

$$\delta(f(x)) = \sum_{x_* \text{ s.t. } f(x_*)=0} \frac{\delta(x - x_*)}{|f'(x_*)|} \tag{2.78}$$

where $f'(x)$ denotes the derivative of f with respect to its argument. Because

$$\frac{\partial}{\partial E}\Big[E^2 - |\vec{p}|^2 - m^2\Big] = 2E \tag{2.79}$$

and $E^2 - \vec{p}^{\,2} - m^2 = 0$ when $E = \pm E_{\vec{p}} \equiv \pm\sqrt{m^2 + \vec{p}^{\,2}}$,

$$\delta(p^2 - m^2) = \frac{1}{2\sqrt{m^2 + |\vec{p}|^2}}\Big[\delta(E - E_{\vec{p}}) + \delta(E + E_{\vec{p}})\Big] \tag{2.80}$$

When we substitute this result into Eq. (2.74), the Heaviside step function $\theta(E)$ causes the negative energy term $\propto \delta(E + E_{\vec{p}})$ to vanish, such that

$$\int \frac{d^3p}{(2\pi)^3} \; \frac{1}{2E_{\vec{p}}} \; f(E_{\vec{p}}, \vec{p}) \tag{2.81}$$

which is a 3-momentum integral as desired. Because the original integral is Lorentz invariant, this expression must be as well. The integration weight factor $d^3p/[(2\pi)^3 2E_{\vec{p}}]$ will occur frequently in definitions and calculations due to its Lorentz invariance.

When calculating quantities involving n particles (labeled 1, 2, ..., n) with individual 4-momenta $p_i = (E_i, \vec{p}_i)$ that are constrained to have some total 4-momentum P (but otherwise unconstrained), integrals of the form

$$\int \left[\prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3} \; \frac{1}{2E_{\vec{p}_i}} \right] \left[(2\pi)^4 \delta^4 \left(P - \sum_{i=1}^{n} p_i \right) \right] f(E_{\vec{p}_1}, \vec{p}_1, \cdots, E_{\vec{p}_n}, \vec{p}_n) \tag{2.82}$$

regularly occur, where the first bracketed factor is called the n-particle Lorentz-invariant phase space element,

$$d\Pi_n = \prod_{i=1}^{n} \left[\frac{d^3p_i}{(2\pi)^3} \frac{1}{2E_{\vec{p}_i}} \right] \tag{2.83}$$

In the following section, the Poincaré generators are promoted to their quantum equivalents in preparation of defining external particle states with well-defined 4-momentum and helicity.

2.3 Poincaré Group: Quantum Promotion

2.3.1 Quantum Mechanics

In the previous section, demanding a universal speed of light motivated us to investigate the group of linear transformations that left the inner products $p \cdot q = \eta_{\mu\nu} p^\mu q^\mu$ unchanged. This led us to the Lorentz group, which combines rotations and boosts, and its generalization the Poincaré group, which additionally incorporates spacetime translations. The different transformations in the Poincaré group map between reference frames while globally preserving the speed of light.

In the present section, we extend these ideas to quantum mechanics. However, whereas our investigation of 4-vector transformations was motivated by the frame independence of the speed of light, the promotion to quantum mechanics is motivated by the frame independence of experimental outcomes. To be concrete: while observers in different reference frames will disagree about their spacetime coordinates by a Poincaré transformation, once those differences are accounted for they should agree on—for example—how many heads or tails are measured in a sequence of coin flips. Consequently, so long as our experimental questions are phrased in frame-independent ways, the related experimental probabilities should be frame-independent as well.

Quantum mechanical states are described via kets $|\psi\rangle$. Two kets describe identical realities if they differ at most by a phase, e.g. $|\psi\rangle$ and $e^{i\alpha}|\psi\rangle$ correspond to physically-indistinguishable systems for any real choice of α. A complete set of kets is defined for a system by choosing a maximally-commuting set of observables for that system, whether the observables are described by self-adjoint operators (A such that $A^\dagger = A$). From here, we will use the descriptors "self-adjoint" and "Hermitian" interchangeably. A complete set of kets defines a Hilbert space. Defining an orthonormality condition on a complete set of kets implies a complete set of bras $\langle\psi|$ as well as a resolution of identity on the space. This defines an inner product between bras and kets which satisfies $\langle\psi_1|\psi_2\rangle^* = \langle\psi_2|\psi_1\rangle$ for any two kets $|\psi_1\rangle$ and ψ_2. The probability (or probability density) associated with measuring a state ψ as another state ψ' is

$$\text{Prob}(\psi \to \psi') \equiv |\langle\psi|\psi'\rangle|^2 \tag{2.84}$$

where it is assumed the kets are normalized.

A symmetry transformations A on a Hilbert space is any transformation which preserves probabilities, i.e. if $|\psi_1\rangle$ and $|\psi_2\rangle$ are arbitrary kets in the Hilbert space and are transformed such that $|\psi_1\rangle \to |A\psi_1\rangle = A|\psi\rangle$ and $|\psi_2\rangle \to A|\psi_2\rangle$, then A is a symmetry transformation if

$$|\langle\psi_1|\psi_2\rangle|^2 \quad \mapsto \quad |\langle A\psi_1|A\phi_2\rangle|^2 = |\langle\psi_1|A^\dagger A|\phi_2\rangle|^2 = |\langle\psi_1|\phi_2\rangle|^2 \tag{2.85}$$

Wigner's Theorem establishes that a symmetry transformation A must either be unitary and linear,

$$\langle A\psi_1|A\psi_2\rangle = \langle\psi_1|\psi_2\rangle \qquad \text{and} \qquad A\Big[c_1|\psi_1\rangle + c_2|\psi_2\rangle\Big] = c_1|A\psi_1\rangle + c_2|A\psi_2\rangle \tag{2.86}$$

or antiunitary and antilinear,

$$\langle A\psi_1|A\psi_2\rangle = \langle\psi_1|\psi_2\rangle^* \qquad \text{and} \qquad A\Big[c_1|\psi_1\rangle + c_2|\psi_2\rangle\Big] = c_1^*|A\psi_1\rangle + c_2^*|A\psi_2\rangle \tag{2.87}$$

where c_1 and c_2 are complex numbers. If A is unitary, then $\tilde{A} = A^\dagger$.

Suppose there exists a group of real transformations $\{A\}$ (like the 4-vector representation of the Poincaré group) where each transformation is continuously connected to the identity such that each transformation A can be expressed as exponentiations of real generators G_a

$$A(\xi) = \text{Exp}\Big[\sum_a \xi_a G_a\Big] \tag{2.88}$$

via real parameters ξ_a, where the generators satisfy some commutation relations

$$[G_a, G_b] = \sum_c T_{abc}\, G_c \tag{2.89}$$

for some real numbers T_{abc}. In the quantum theory, we can recreate the action of the set $\{A\}$ on our kets by mapping each transformation A to a unitarity operator $\mathcal{U}[A]$ of the exponentiated form

$$\mathcal{U}[A(\xi)] = \text{Exp}\Big[-i\sum_a \xi_a \mathcal{H}[G_a]\Big] \tag{2.90}$$

where the Hermitian operators $\mathcal{H}[G_a]$ satisfy the commutation relations

$$\Big[\mathcal{H}[G_a], \mathcal{H}[G_b]\Big] = \sum_c i\, T_{abc}\, \mathcal{H}[G_c] \tag{2.91}$$

for those *same* real numbers T_{abc}. Heuristically, Eq. (2.89) goes to (2.91) by replacing the generators G_a with $-i\mathcal{H}[G_a]$. The operators $\mathcal{H}[G_a]$ are also called generators, although in this case they are generators of the unitary operators $\mathcal{U}[A]$. When context is sufficient (and to minimize clutter), we will simply write G_a in place of $\mathcal{H}[G_a]$. If the original transformation is active (passive), then the resulting quantum operator will encode an active (passive) transformation as well. Recall that our generators from the previous section were derived for the active transformations.

2.3.2 Promoting the Poincaré Generators

The symmetry group of spacetime is the Poincaré group, the generators of which were previously found to satisfy various commutation relations, Eqs. (2.71)-(2.73). We now promote each of those generators to Hermitian operators as to create unitary representations of the corresponding Poincaré transformations, i.e. the matrices $\{P^\mu, J^i, K^i\}$ will be mapped to operators $\{\mathcal{H}[P^\mu], \mathcal{H}[J^i], \mathcal{H}[K^i]\}$. Note that this mapping is not unique: different particles even within the same scenario often require different choices of Hermitian generators. However, whatever Hermitian generators we choose for a particular representation, they must satisfy the promoted version of the previously-derived commutation structure:

$$[J^i, J^j] = +i\epsilon_{ijk}J^k \qquad [J^i, K^j] = +i\epsilon_{ijk}K^k \qquad [K^i, K^j] = -i\epsilon_{ijk}J^k \tag{2.92}$$

$$[H, J^i] = 0 \qquad [H, K^i] = +iP^i \qquad [J^i, P^j] = +i\epsilon_{ijk}P^k \qquad [P^i, K^j] = +iH\delta_{ij} \tag{2.93}$$

$$[P^\mu, P^\nu] = 0 \tag{2.94}$$

where we have dropped the $\mathcal{H}$ label and have been cautious of the minus sign present in the exponentiation of the time-translation generator H (as in Eq. (2.65)). The operator H is the Hamiltonian, and an eigenket of H with eigenvalue E is said to have energy E.

Utilizing these generators, we obtain unitary operators that apply the effect of a generic rotation, boost, or translation to a ket:

$$\mathcal{U}[R(\vec{\alpha})] = \text{Exp}\Big[-i\vec{\alpha}\cdot\vec{J}\Big] \qquad \mathcal{U}[B(\vec{\beta})] = \text{Exp}\Big[-i\vec{\beta}\cdot\vec{K}\Big] \qquad \mathcal{U}[T(\epsilon)] = \text{Exp}\Big[+i(\epsilon\cdot P)\Big] \tag{2.95}$$

The operators $\vec{J}$ and $\vec{P}$ are the angular momentum and (linear) momentum operators respectively, and the rotation Casimir operator $\vec{J}^2$ is the total angular momentum operator. In going over to the quantum equivalent, we inadvertently expand the group structure of our spacetime symmetries. For example, the Lie algebras $\mathfrak{so}(3)$ (associated with the rotation group $\mathbf{SO(3)}$) and $\mathfrak{su}(2)$ (associated with $\mathbf{SU(2)}$, the universal covering group of $\mathbf{SO(3)}$) actually have identical commutation relations as far as quantum generators are concerned. Because the quantum operators are only restricted by the commutation relations in Eq. (2.91), we are able to represent 4-vector rotations (a representation of $\mathbf{SO(3)}$) as unitarty representations of $\mathbf{SU(2)}$. Irreducible unitary representations of $\mathbf{SU(2)}$ are reviewed in Section 2.6.

As mentioned above, the time translation operator $H \equiv P^0$ is identified as the Hamiltonian, and yields a time evolution operator $\mathcal{U}(\Delta t)$,

$$\mathcal{U}(\Delta t) = \text{Exp}\left[-i\,(\Delta t)\,H \right] \tag{2.96}$$

Note the minus sign in the exponent relative to the time translation operator in Eq. (2.95). This is consistent with the discussion in Subsection 2.2.2.

2.3.3 The Square of the 4-Momentum Operator

There are several important combinations of the generators relevant to our calculation. The first is square of the 4-momentum operator $P^2 \equiv H^2 + \vec{P}^{\,2}$, which is important because of its ties to particle mass. In particular, a state $|\psi\rangle$ has mass $M \geq 0$ if $P^2|\psi\rangle = M^2|\psi\rangle$. Because P^2 is a Casimir operator of the Poincaré group, all of the generators automatically commute with it. If a single-particle state is simultaneously an eigenstate of P^2 and H, then it is automatically also an eigenstate of the total 3-momentum operator $\vec{P}^{\,2}$, and we can choose to label (and normalize) those states with either their energy or 3-momentum magnitude.

2.3.4 The Helicity Operator

Another important operator formed by combining Poincaré generators is the helicity operator Λ. However, before we define the helicity operator, let us instead consider a related operator: the inner product $\vec{J}\cdot\vec{P} = J^1P^1 + J^2P^2 + J^3P^3$.

The operator $\vec{J}\cdot\vec{P}$ commutes with many of the Poincaré generators. For example, because $[AB, C] = [A, C]B + A[B, C]$ and $[P^i, P^j] = 0$ (and recalling $P_z = P^3$),

$$\begin{aligned}
[P_z, \vec{J}\cdot\vec{P}] &= \sum_{i=1}^{3}[P^3, J^i]P^i + J^i[P^3, P^i] && (2.97)\\
&= [P^3, J^1]P^1 + [P^3, J^2]P^2 && (2.98)\\
&= iP^2P^1 - iP^1P^2 && (2.99)\\
&= 0 && (2.100)
\end{aligned}$$

such that $[P^i, \vec{J}\cdot\vec{P}] = 0$ for all i via cyclic symmetry and thereby $[\vec{P}^{\,2}, \vec{J}\cdot\vec{P}] = 0$ as well. Similarly,

$$\begin{aligned}
[J_z, \vec{J}\cdot\vec{P}] &= \sum_{i=1}^{3}[J^3, J^i]P^i + J^i[J^3, P^i] && (2.101)\\
&= [J^3, J^1]P^1 + J^1[J^3, P^1] + [J^3, J^2]P^2 + J^2[J^3, P^2] && (2.102)\\
&= iJ^2P^1 + iJ^1P^2 - iJ^1P^2 - iJ^2P^1 && (2.103)\\
&= 0 && (2.104)
\end{aligned}$$

such that $[J^i, \vec{J}\cdot\vec{P}] = 0$ via cyclic symmetry, and $[\vec{J}^2, \vec{J}\cdot\vec{P}] = 0$ as well. Finally, note that

$$[H, \vec{J}\cdot\vec{P}] = \sum_{i=1}^{3}[H, J^i]P^i + J^i[H, P^i] = 0 \tag{2.105}$$

vanishes too, because $[H, P^i] = [H, J^i] = 0$. Hence, in all, $\vec{J}\cdot\vec{P}$ commutes with H, P^i, $\vec{P}^2$, J^i, and $\vec{J}^2$.

Suppose we restrict our attention to eigenkets $|EM\rangle$ of the Hamiltonian H and total 4-momentum operator P^2, e.g.

$$H|Em\rangle = E|EM\rangle \qquad P^2|Em\rangle = m^2|Em\rangle \tag{2.106}$$

where $E > 0$ and $M \geq 0$ are the associated state energy and mass respectively. All of the single-particle states that we consider have well-defined energy and mass in this way. For these states, we define the helicity operator Λ as

$$\Lambda \equiv \frac{\vec{J}\cdot\vec{P}}{\sqrt{E^2 - M^2}} \tag{2.107}$$

which is pivotal to defining the external states relevant to this dissertation. Like the operator $\vec{J}\cdot\vec{P}$ to which it is proportional, Λ commutes with P^i, $\vec{P}^2$, J^i, and $\vec{J}^2$.

When describing external single-particle states, we will consider the relation of two maximally-commuting sets of observable operators, both of which involve the helicity operator:

- **Option 1:** P^μ, Λ
- **Option 2:** H, $\vec{J}^2$, J_z, Λ

The single-particle states will also have definite masses and spins, and thus be eigenkets of the corresponding operators; however, because they are associated with Casimir operators of the Poincaré group, we can (and will) always include them in our maximally-commuting set. As such, we will not label our single-particle states with mass or spin after this point. Helicity eigenstates will be considered in more detail in Section 2.7.

2.3.5 Aside: Lorentz Group as $SU(2)\times SU(2)$

Although the boost generators mix with the rotation generators under commutation according to Eqs. (2.56) and (2.57), there is a trick for disentangling the two sets. To do so, we must go over to the complexification of the Lorentz group, wherein we will treat the following combinations of the rotation and boost generators as Hermitian:

$$\vec{\mathcal{A}} \equiv \frac{1}{2}(\vec{J} + i\vec{K}) \qquad \vec{\mathcal{B}} \equiv \frac{1}{2}(\vec{J} - i\vec{K}) \tag{2.108}$$

However, this assumption necessarily implies that (contrary to our previous construction) our boost generators must be anti-Hermitian:

$$\vec{J} = \vec{\mathcal{A}} + \vec{\mathcal{B}} \qquad \vec{K} = -i(\vec{\mathcal{A}} - \vec{\mathcal{B}}) \tag{2.109}$$

This is a nontrivial price to pay, but it comes at a great benefit: as promised, the commutators of the Lorentz generators decouple

$$[A_i, A_j] = +\epsilon_{ijk} A_k \qquad [A_i, B_j] = 0 \qquad [B_i, B_j] = +\epsilon_{ijk} B_k \tag{2.110}$$

In fact, the operators described by $\{A^i\}$ and $\{B^i\}$ follow the same Lie algebra structure as the angular momentum operators $\{J^i\}$. In this sense, the complexified Lorentz group is two independent copies of $\mathbf{SU(2)}$. Because each $\mathbf{SU(2)}$ admits finite-dimensional unitary representations, complexification allows us to construct finite-dimensional unitary representations of the Lorentz group too (which is not possible prior to complexification). Furthermore, because combining $\mathbf{SU(2)}$ representations yields another $\mathbf{SU(2)}$ representation, the equation $\vec{J} = \vec{\mathcal{A}} + \vec{\mathcal{B}}$ ensures that the end product behaves in a well-defined way under the rotation group.

2.4 External States and Matrix Elements

2.4.1 Single-Particle States: Definite 4-Momentum

In quantum mechanics, the kets describing physical states are chosen to span the eigenvalues of certain Hermitian operators corresponding to observable quantities. Specifically, given a commuting set of observables $\{A_1, \cdots A_N\}$ (so that $[A_i, A_j] = 0$ for any pair A_i, A_j), we can form a complete set of kets $\{|a_1 \cdots a_n\rangle\}$ where

$$A_i |a_1 \cdots a_n\rangle = a_i |a_1 \cdots a_n\rangle \tag{2.111}$$

for each $i \in \{1, \cdots, n\}$. Because each operator A_i is Hermitian, each eigenvalue a_i is real. The resulting collection of kets form a complete basis and are equipped with a convention-dependent orthonormalization condition.

For the duration of this chapter, we use (interaction picture) kets to describe the initial and final multi-particle states of scattering processes, each of which is built from direct products of single-particle states. Thus, we turn our focus to the construction of single-particle states. Following Wigner's classification [19], our single-particle states are chosen to be unitary irreducible representations of the Poincaré group, and will have definitive mass and total spin (or helicity, if massless). We will choose choose these states so that they have well-defined 4-momentum (and eventually helicity, as detailed in Section 2.7). We can choose 4-momentum as quantum numbers because the 4-momentum operators of Subsection 2.3.2 form a commuting set ($[P^\mu, P^\nu] = 0$ for all $\mu, \nu \in \{0, 1, 2, 3\}$) and the 4-momentum operators encode an observable. Because the energy eigenvalue E associated with the Hamiltonian H is constrained by the particle's mass m to satisfy $E^2 = m^2 + \vec{p}^{\,2}$, we only label the kets with the 3-momentum eigenvalues, i.e. as $|\vec{p}\rangle \equiv |p_x\, p_y\, p_z\rangle$. By definition, these satisfy

$$H|\vec{p}\rangle = \sqrt{m^2 + |\vec{p}|^2}\, |\vec{p}\rangle \qquad \vec{P}|\vec{p}\rangle = \vec{p}\, |\vec{p}\rangle \tag{2.112}$$

where we recall that $H \equiv P^0$. Because 3-momentum is a continuous degree of freedom, these kets are normalized by a Dirac delta, such that

$$\langle \vec{p} | \vec{p}\,' \rangle \propto \delta^3(\vec{p} - \vec{p}\,') \tag{2.113}$$

up to some proportionality factor. The exact choice of this proportionality factor varies throughout the literature. We motivate our particular choice via the Lorentz-invariant phase space element derived in Subsection 2.2.6. Namely, we would like to normalize our kets such that we can resolve the identity on this space via an integral weighted by the Lorentz-invariant factor $d^3\vec{p}/[(2\pi)^3 2E_{\vec{p}}]$:

$$\mathbb{1} = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}|\vec{p}\rangle\langle\vec{p}| \tag{2.114}$$

This implies

$$|\vec{k}\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}|\vec{p}\rangle\langle\vec{p}|\vec{k}\rangle \tag{2.115}$$

which achieved as long as we choose our normalization such that

$$\langle\vec{p}|\vec{p}\,'\rangle = (2\pi)^3\,(2E_{\vec{p}})\,\delta^3(\vec{p}-\vec{p}\,') \tag{2.116}$$

and so we do. A simultaneous eigenstate of P^2 and H is also an eigenstate of $\vec{P}^{\,2}$, so we could use $|\vec{p}|$ instead of E as a quantum number. On the occasion we would like to do so, we define an alternate collection of 3-momentum kets $||\vec{p}|\theta\phi\rangle$, which are expressed in spherical coordinates and normalized via

$$\langle|\vec{p}|\theta\phi||\vec{p}\,'|\theta'\phi'\rangle = \frac{1}{|\vec{p}|^2}\,\delta(|\vec{p}|-|\vec{p}\,'|)\,\delta^2(\Omega-\Omega') \tag{2.117}$$

where

$$\delta^2(\Omega-\Omega') \equiv \delta(\phi-\phi')\,\delta(\cos\theta-\cos\theta') \tag{2.118}$$

and $\theta,\theta' \in [0,\pi]$ and $\phi,\phi' \in [0,2\pi)$, such that

$$\mathbb{1} = \int |\vec{p}|^2\,d|\vec{p}|\,d\Omega\;\,||\vec{p}|\theta\phi\rangle\langle|\vec{p}|\theta\phi| \tag{2.119}$$

on this space.

Eq. (2.116) expresses a lot of information about the space of 3-momentum kets, but we can add further structure to this space using our knowledge of spacetime transformations: we know from our considerations of the Lorentz group in Section 2.2.1 that any two 4-momenta on the same mass hyperboloid can be related via a Lorentz transformation. Consequently, given a Lorentz transformation Λ that maps a 4-momentum p to a 4-momentum p', there exists a unitary operator $\mathcal{U}[\Lambda]$ that maps $|\vec{p}\rangle$ to $|\vec{p}\,'\rangle$ up to a phase:

$$|\vec{p}\,'\rangle \propto \mathcal{U}[\Lambda]\,|\vec{p}\rangle \tag{2.120}$$

While it may be tempting to set this to an equality, such an equality would not be well-defined because there are *many* distinct Lorentz transformations that take p to p'. Therefore, to uniquely identify individual kets we follow Wigner's lead [19] and choose a standard 4-momentum k on each Lorentz invariant 4-momentum hypersurface. Then, for every other 4-momentum p on a given hypersurface, we choose a standard Lorentz transformation that maps the corresponding standard 4-momentum k to $\vec{p}$. By choosing these standard 4-momentum and transformations, we eliminate the ambiguity of the above proportionality and can establish a well-defined equality.

The details of these standards depend on the mass of the single-particle state in question:

- **Massive:** For a single-particle state with mass $m > 0$, we choose the rest frame 4-momentum $k^\mu = (m, \vec{0})$. To obtain any other 4-momentum p having equal mass, we first boost along z until it has 3-momentum $|\vec{p}|\hat{z}$ and then rotate via $R(\theta, \phi)$ to attain a 3-momentum $\vec{p}$. This allows us to define, unambiguously,

$$|\vec{p}\rangle = \mathcal{U}[R(\phi,\theta)]\,\mathcal{U}[B_z(\beta_{k\to p})]\,|\vec{0}\rangle \tag{2.121}$$

  where $\beta_{k\to p} = \text{arccosh}(E_{\vec{p}}/m)$.

- **Massless:** For a single-particle state with vanishing mass $m = 0$, there is no rest frame, so instead we choose a standard light-like 4-momentum $(E_{\vec{k}}, E_{\vec{k}}\hat{z})$ for some choice of energy $E_{\vec{k}}$. From here the procedure mimics the massive case: to obtain any other 4-momentum p on the light cone, we first boost along z until it has 3-momentum $|\vec{p}|\hat{z}$ and then rotate via $R(\theta, \phi)$ to attain a 3-momentum $\vec{p}$. This allows us to define, unambiguously,

$$|\vec{p}\rangle = \mathcal{U}[R(\phi,\theta)]\,\mathcal{U}[B_z(\beta_{k\to p})]\,|\vec{k}\rangle \tag{2.122}$$

  where now $\beta_{k\to p} = \ln(E_{\vec{p}}/E_{\vec{k}})$.

We will revisit these procedures when constructing helicity eigenstates in Section 2.7.

The above discussion glosses over an important (but ultimately inconsequential) technicality. The only physical states are those which have finite normalizations. Because the 3-momentum kets are normalized to a Dirac delta, they are unphysical, and thus in principle cannot serve as external states in physical scattering processes. This reflects the fact that we cannot in practice construct a system with definite 3-momentum. Even in the most ideal of experimental conditions, the existence of such a state is forbidden by the Heisenberg uncertainty principle. Therefore, we should actually perform calculations in quantum field theory using wavepacket superpositions of states. For example, rather than using a ket $|\vec{p}\,\rangle$, we might instead use the wavepacket

$$|\psi_{\vec{p}}\rangle = \int \frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{2E_{\vec{q}}}\;\psi_{\vec{p}}(\vec{q}\,)\,|\vec{q}\rangle \tag{2.123}$$

where $\psi_{\vec{p}}(\vec{q})$ is a three-dimensional Gaussian sharply peaked as $\vec{q} = \vec{p}$. The smoothing this wavepacket provides is sufficient to yield a finite normalization:

$$\langle\psi_{\vec{p}}|\psi_{\vec{p}}\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{2E_{\vec{k}}}\frac{1}{2E_{\vec{q}}}\;\psi^*_{\vec{p}}(\vec{k}\,)\,\psi_{\vec{p}}(\vec{q}\,)\,\langle\vec{k}|\vec{q}\rangle \tag{2.124}$$

$$= \int \frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{2E_{\vec{q}}}\;|\psi_{\vec{p}}(\vec{q}\,)|^2 \tag{2.125}$$

This is important when deriving results like the LSZ reduction formula (which relates external states to quantum fields), but as far as matrix elements are concerned we can always take the limit as the wavepacket becomes a Dirac delta and thereby use the 3-momentum kets as external states (even if technically we should not). Because this dissertation only calculates matrix elements and does not derive results sensitive to this technicality, it will be ignored.

2.4.2 Multi-Particle States: Definite 4-Momentum

A multi-particle state composed of single-particle states will have well-defined mass, total spin, and 4-momentum for each particle included in the state. As such, we can construct a space of n-particle kets labeled $|\vec{p}_1\, \vec{p}_2\, \cdots\, \vec{p}_n\rangle$ where each 3-momentum $\vec{p}_i$ labels a particle with definite mass m_i. In the absence of identical particles (more on that soon), the n-particle states can be constructed (up to a conventional phase, per usual) by taking the direct product of n single-particle states:

$$|\vec{p}_1\, \vec{p}_2\, \cdots\, \vec{p}_n\rangle \propto |\vec{p}_1\rangle \otimes |\vec{p}_2\rangle \otimes \cdots \otimes |\vec{p}_n\rangle \tag{2.126}$$

These states are assumedly arranged in some canonical ordering based on their distinguishability, e.g. electrons are listed left of muons and so-on, and electron kets vanish when contracted with muon bras. We choose the free phase in Eq. (2.126) to be +1 so that equality replaces the proportionality. However, regardless of the particular phase selected, the multi-particle normalization is implied by the single-particle normalization Eq. (2.116). By complex squaring both sides of Eq. 2.126, the multi-particle normalization is found to equal

$$\langle\vec{p}_1\, \vec{p}_2\, \cdots\, \vec{p}_n|\vec{k}_1\, \vec{k}_2\, \cdots\, \vec{k}_n\rangle = \prod_{i=1}^{n} (2\pi)^3\, (2E_{\vec{p}_i})\, \delta^3(\vec{p}_i - \vec{k}_i) \tag{2.127}$$

from which the n-particle resolution of identity (without identical particles) equals

$$\mathbb{1} = \int \prod_{i=1}^{n} \left[\frac{d^3p_i}{(2\pi)^3}\frac{1}{2E_{p_i}}\right]\; |\vec{p}_1\vec{p}_2\cdots\vec{p}_n\rangle\langle\vec{p}_1\vec{p}_2\cdots\vec{p}_n| \tag{2.128}$$

Note the presence of the n-particle Lorentz-invariant phase space measure. The above construction is sufficient if all particles are distinguishable. In that case, we can imagine an additional indicator being added to each 3-momentum label in the ket that gives a unique name to each particle beyond its 3-momentum content. Then, when we perform the inner product described in Eq. (2.127), we could pair up particles in the bra and ket based on matching their names to obtain the correct Dirac deltas (and if we cannot find such a collection of pairs then we know the inner product vanishes). If any number of the particles involved are instead identical, then we must be more careful in our construction of the ket space.

Two particles are identical if they share all of the same intrinsic quantum numbers—such as mass, total spin, and gauge transformation properties—and a particular set of such properties defines a particle species. For example, as listed in Figure 1.2, the particle species known as “top quark” is characterized by a mass of 173 GeV, total spin $\frac{1}{2}$, electric charge $+\frac{2}{3}$, and triplet transformation behavior under the color gauge group $\mathbf{SU(3)_C}$. Because they are spin-$\frac{1}{2}$ particles, each top quark can be measured as either spin up ($m = +\frac{1}{2}$) or spin down ($m = -\frac{1}{2}$) with respect to a given projection axis; however, the need for a projection axis indicates that although projected spin is an *internal* quantum number, it is not an *intrinsic* quantum number. Thus, spin up and spin down top quarks are still regarded as identical in the technical sense. This applies to color charge as well: the status of a top quark as red, green, or blue (or a specific superposition of those colors) is a gauge-dependent quality, and so color charge is not an intrinsic quantum number. (This contrasts with electric charge, which *does* possess a gauge-independent value.) Meanwhile, despite a charm quark possessing nearly all of the same intrinsic quantum numbers as the top quark, the two quarks differ in mass and thus are distinguishable regardless of further details.

To demonstrate that the existing machinery is insufficient for the construction of multi-particle states involving identical particles, suppose we try to use the previous construction to describe a 2-particle state consisting of identical particles with distinct 3-momenta $\vec{p}_1$ and $\vec{p}_2$. If the previous construction truly is sufficient, then (because the particles are identical) the kets $|\vec{p}_1\vec{p}_2\rangle$ or $|\vec{p}_1\vec{p}_2\rangle$ describe indistinguishable physical realities and thus must be equal up to a phase χ:

$$|\vec{p}_1\vec{p}_2\rangle \stackrel{?}{=} \chi|\vec{p}_2\vec{p}_1\rangle \tag{2.129}$$

If we swap the order of the labels in the RHS ket once more (and assume χ is agnostic to the details of the 3-momenta encoded by the ket[1]), then we return to the original ordering and gain another factor of χ

$$|\vec{p}_1\vec{p}_2\rangle \stackrel{?}{=} \chi^2|\vec{p}_1\vec{p}_2\rangle \tag{2.130}$$

where equality only holds true if $\chi^2 = 1$. Note that χ^2 is a regular square (i.e. not a complex square), so this restricts χ to equaling $+1$ or -1. The exact choice of one sign over the other is an intrinsic property of the particle being considered and is ultimately tied to the spin of the given particle. Unfortunately, Eq. (2.129) is inconsistent with the normalization defined in Eq. (2.127): specifically,

$$0 = \langle\vec{p}_1\,\vec{p}_2|\vec{p}_2\,\vec{p}_1\rangle \stackrel{?}{=} \chi\langle\vec{p}_1\,\vec{p}_2|\vec{p}_1\,\vec{p}_2\rangle = \chi\prod_{i=1}^{2}(2\pi)^3\,(2E_{\vec{p}_i})\,\delta^3(0) \tag{2.131}$$

which is zero on the LHS, but infinite on the RHS. The origin of this obstruction lies in Eq. (2.126), where we expressed an n-particle ket as a direct product of single particle kets. The ordering in the direct product $|\vec{p}_1\rangle \otimes |\vec{p}_2\rangle$ is absolute and lacks the exchange symmetry we desire, e.g.

$$|\vec{p}_1\rangle \otimes |\vec{p}_2\rangle \neq \chi|\vec{p}_2\rangle \otimes |\vec{p}_1\rangle \tag{2.132}$$

[1] This is a nontrivial assumption. Thankfully, even when this assumption is dropped one can still recover the same end result, although doing so requires a good amount of homotopy theory to demonstrate that the 3-momentum-dependent phase is always removable via ket redefinitions.

To remedy this, we define the following symmetric and antisymmetric kets:

$$|\vec{p}_1\,\vec{p}_2\,\cdots\,\vec{p}_n) = \frac{1}{\sqrt{n!}}\left[\prod_{i=1}^{N}\frac{1}{\sqrt{n_i!}}\right]\sum_{\pi\in\pi_n}|\pi(\vec{p}_1,\vec{p}_2,\cdots,\vec{p}_n)\rangle \tag{2.133}$$

$$|\vec{p}_1\,\vec{p}_2\,\cdots\,\vec{p}_n] = \frac{1}{\sqrt{n!}}\sum_{\pi\in\pi_n}\text{sign}(\pi)\,|\pi(\vec{p}_1,\vec{p}_2,\cdots,\vec{p}_n)\rangle \tag{2.134}$$

where π_n denotes the set of all n-element permutations and sign(π) refers to the parity of a permutation π ($+1$ for even permutations, -1 for odd permutations). The exact prefactors in front of each permutation sum are chosen to guarantee upcoming normalization formulae (Eqs. (2.144) and (2.145)). Within the symmetrized case in particular, care must be taken to account for potential repeats of particle information, e.g. (because we continue to neglect other quantum numbers) when two identical particles have identical 3-momentum $\vec{p}_1 = \vec{p}_2$. To be explicit, suppose among the n particle labels there is only N unique labels present. The n_i present in Eq. (2.133) takes into account possible label repeats and equals how many times a given unique label occurs in the list $(\vec{p}_1\,,\,\vec{p}_2\,,\,\cdots\,,\,\vec{p}_n)$. Thus, $n = n_1 + \cdots + n_N$. For future use, it is useful to define a symbol $\mathcal{S}(\vec{p}_1,\vec{p}_2,\cdots,\vec{p}_n)$ for this repeated label information:

$$\mathcal{S}(\vec{p}_1,\vec{p}_2,\cdots,\vec{p}_n) \equiv \prod_{i=1}^{N} n_i! \tag{2.135}$$

where n_i and N are defined for the list $(\vec{p}_1,\vec{p}_2,\cdots,\vec{p}_n)$ in the same way as they are defined in the preceding paragraph. The identical particle kets defined in Eqs. (2.133) and (2.134) are fully symmetric and antisymmetric in their particle labeling respectively: that is, given a permutation π, they satisfy

$$|\pi(\vec{p}_1\,\vec{p}_2\,\cdots\,\vec{p}_n)) = |\vec{p}_1\,\vec{p}_2\,\cdots\,\vec{p}_n) \tag{2.136}$$

$$|\pi(\vec{p}_1\,\vec{p}_2\,\cdots\,\vec{p}_n)] = \text{sign}(\pi)\,|\vec{p}_1\,\vec{p}_2\,\cdots\,\vec{p}_n] \tag{2.137}$$

Particles described by the multi-particle symmetrized kets ($\chi = +1$) are bosons and particles described by the multi-particle antisymmetrized kets ($\chi = -1$) are fermions [20, 21]. The antisymmetry of the latter kets is why we need not worry about repeated labels when normalizing that case; if any labels are repeated (e.g. two particles have identical quantum numbers, which at present means identical 3-momenta), then the ket will automatically vanish:

$$|\vec{p}\,\vec{p}\,\vec{p}_3\,\cdots\,\vec{p}_n] = -|\vec{p}\,\vec{p}\,\vec{p}_3\,\cdots\,\vec{p}_n] \qquad \Longrightarrow \qquad |\vec{p}\,\vec{p}\,\vec{p}_3\,\cdots\,\vec{p}_n] = 0 \tag{2.138}$$

This is an expression of the Pauli exclusion principle [21], which states that identical fermions are forbidden from having fully identical quantum numbers.

We now address the normalizations of these identical particle states. For multi-particle states composed of a bosonic species,

$$(\vec{p}_1\,\vec{p}_2\,\cdots\,\vec{p}_n|\vec{p}_1'\,\vec{p}_2'\,\cdots\,\vec{p}_n') = \frac{1}{n!}\left[\prod_{i=1}^{N}\frac{1}{\sqrt{n_i!}}\right]\left[\prod_{j=1}^{N'}\frac{1}{\sqrt{n_j'!}}\right] \times \sum_{\pi\in\pi_n}\sum_{\pi'\in\pi_n}\langle\pi(\vec{p}_1,\vec{p}_2,\cdots,\vec{p}_n)|\pi'(\vec{p}_1',\vec{p}_2',\cdots,\vec{p}_n')\rangle \tag{2.139}$$

$$= \frac{1}{n!}\left[\prod_{i=1}^{N}\frac{1}{\sqrt{n_i!}}\right]\left[\prod_{j=1}^{N'}\frac{1}{\sqrt{n_j'!}}\right] n!\sum_{\pi\in\pi_n}\langle\pi(\vec{p}_1,\vec{p}_2,\cdots,\vec{p}_n)|\vec{p}_1'\,\vec{p}_2'\,\cdots\,\vec{p}_n'\rangle \tag{2.140}$$

$$= \frac{1}{n!}\left[\prod_{i=1}^{N}\frac{1}{\sqrt{n_i!}}\right]\left[\prod_{j=1}^{N'}\frac{1}{\sqrt{n_j'!}}\right] n!\left[\prod_{i=1}^{N}n_i!\right]\sum_{\text{unique }\pi\in\pi_n}\langle\pi(\vec{p}_1,\vec{p}_2,\cdots,\vec{p}_n)|\vec{p}_1'\,\vec{p}_2'\,\cdots\,\vec{p}_n'\rangle \tag{2.141}$$

$$= \left[\prod_{i=1}^{N}\sqrt{n_i!}\right]\left[\prod_{j=1}^{N'}\frac{1}{\sqrt{n_j'!}}\right]\sum_{\text{unique }\pi\in\pi_n}\langle\pi(\vec{p}_1,\vec{p}_2,\cdots,\vec{p}_n)|\vec{p}_1'\,\vec{p}_2'\,\cdots\,\vec{p}_n'\rangle \tag{2.142}$$

where "unique $\pi\in\pi_n$" means only summing over a subset of permutations π that yield unique lists $\pi(\vec{p}_1,\vec{p}_2,\cdots,\vec{p}_n)$. Consequently, if there is no permutation π such that $\pi(\vec{p}_1,\vec{p}_2,\cdots,\vec{p}_n) = (\vec{p}_1',\vec{p}_2',\cdots,\vec{p}_n')$, then the RHS vanishes. However, if such a permutation π does exist, then $N=N'$, $\{n_i\}=\{n_j'\}$, and

$$(\vec{p}_1\,\vec{p}_2\,\cdots\,\vec{p}_n|\vec{p}_1'\,\vec{p}_2'\,\cdots\,\vec{p}_n') = \prod_{i=1}^{n}(2\pi)^3\,(2E_{\vec{p}_i})\,\delta^3(0) \tag{2.143}$$

Therefore, returning to the general case,

$$(\vec{p}_1\,\vec{p}_2\,\cdots\,\vec{p}_n|\vec{p}_1'\,\vec{p}_2'\,\cdots\,\vec{p}_n') = \sum_{\text{unique }\pi\in\pi_n}\langle\pi(\vec{p}_1,\vec{p}_2,\cdots,\vec{p}_n)|\vec{p}_1'\,\vec{p}_2'\,\cdots\,\vec{p}_n'\rangle \tag{2.144}$$

which is the normalization we would have obtained from distinguishable particles. For multi-particle states composed of a fermionic species, the procedure is similar, except that no labels in the bra nor ket may be repeated (or else the inner product automatically vanishes) such that all permutations yield a unique ordering of labels. We must also be cautious of the parity of the permutations involved. After taking these facets into account, we ultimately find

$$[\vec{p}_1\,\vec{p}_2\,\cdots\,\vec{p}_n|\vec{p}_1'\,\vec{p}_2'\,\cdots\,\vec{p}_n'] = \sum_{\pi\in\pi_n}\text{sign}(\pi)\,\langle\pi(\vec{p}_1,\vec{p}_2,\cdots,\vec{p}_n)|\vec{p}_1'\,\vec{p}_2'\,\cdots\,\vec{p}_n'\rangle \tag{2.145}$$

which is again consistent with the normalization we would have obtained from an analogous assortment of distinguishable particles, aside from an overall phase factor (a potential multiplicative -1).

These normalizations imply corresponding resolutions of identity. Let us first consider the bosonic case. To avoid over-counting states, we use the symmetrization of the bosonic kets to arrange the 3-momentum labels in some canonical ordering. The specific canonical ordering is unimportant at present, but one such choice is to rewrite all kets $|\vec{p}_1 \cdots \vec{p}_n)$ so that the 3-momentum are organized from smallest-to-largest in magnitude (with some additional criteria for breaking ties). Whatever the specific choice of canonical ordering, the resulting resolution of identity equals

$$\mathbb{1} = \int_{\text{unique}} \prod_{i=1}^{n} \left[\frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_{p_i}} \right] \; |\vec{p}_1 \vec{p}_2 \cdots \vec{p}_n)(\vec{p}_1 \vec{p}_2 \cdots \vec{p}_n| \tag{2.146}$$

where the "unique" label on the integral indicates that, for instance, if $(\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n) = (\vec{p}^{\,\prime}_1, \vec{p}^{\,\prime}_2, \cdots, \vec{p}^{\,\prime}_n)$ is included in the integral, then no distinct permutation of $(\vec{p}^{\,\prime}_1, \vec{p}^{\,\prime}_2, \cdots, \vec{p}^{\,\prime}_n)$ is also included in the integral. Although in principle this uniquely identifies the bosonic resolution of identity, we would like to rewrite it in a way that does not depend on a specific canonical ordering. To do so, suppose we lift the "unique" label from the RHS of the previous equation so that we integrate over all 3-momentum combinations (regardless if any are related via permutation) and act the resulting operator on a ket $|\vec{k}_1 \vec{k}_2 \cdots \vec{k}_n)$ where all 3-momentum $\vec{k}_i$ are unique. Because $|\vec{k}_1 \vec{k}_2 \cdots \vec{k}_n)$ is symmetric in its labels, it will yield a nonzero result when projected onto any of the bras $(\vec{p}_1 \vec{p}_2 \cdots \vec{p}_n|$ wherein $\pi(\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n) = (\vec{k}_1, \vec{k}_2, \cdots, \vec{k}_n)$ for some permutation π. Because there are $n!$ such permutations,

$$\left[\int \prod_{i=1}^{n} \left[\frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_{p_i}} \right] \; |\vec{p}_1 \vec{p}_2 \cdots \vec{p}_n)(\vec{p}_1 \vec{p}_2 \cdots \vec{p}_n| \right] |\vec{k}_1 \vec{k}_2 \cdots \vec{k}_n) = n! |\vec{k}_1 \vec{k}_2 \cdots \vec{k}_n) \tag{2.147}$$

Therefore, when acting on a ket wherein no set of quantum numbers is repeated,

$$\mathbb{1} = \frac{1}{n!} \int \prod_{i=1}^{n} \left[\frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_{p_i}} \right] \; |\vec{p}_1 \vec{p}_2 \cdots \vec{p}_n)(\vec{p}_1 \vec{p}_2 \cdots \vec{p}_n| \tag{2.148}$$

Of course, the above resolution of identity will not work on a state where there are repeated sets of quantum numbers, because the coincidence of those sets is not as over-counted in the integral. For instance, if $\vec{p}_1 \neq \vec{p}_2$ then the integral over all momentum would catch both $(\vec{p}_1, \vec{p}_2)$ and $(\vec{p}_2, \vec{p}_1)$ despite their equivalence as far as the corresponding symmetrized ket is concerned, whereas if $\vec{p}_1 = \vec{p}_2 = \vec{p}$ then only the single phase space point $(\vec{p}, \vec{p})$ will contribute. Thus, repeated labels yield fewer than $n!$ contributing instances in the integral. When these considerations are generally applied, we obtain a resolution of identity on the whole space of symmetrized kets that does not rely on a specific canonical ordering:

$$\mathbb{1} = \int \prod_{i=1}^{n} \left[\frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_{p_i}} \right] \; \left[\frac{1}{n!} \mathcal{S}(\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n) \right] |\vec{p}_1 \vec{p}_2 \cdots \vec{p}_n)(\vec{p}_1 \vec{p}_2 \cdots \vec{p}_n| \tag{2.149}$$

where $\mathcal{S}$ is defined as in Eq. (2.135). Furthermore, because we will always be acting the bosonic n-particle identity on bosonic n-particle states and

$$(\vec{p}_1 \vec{p}_2 \cdots \vec{p}_n | \vec{k}_1 \vec{k}_2 \cdots \vec{k}_n) = \langle \vec{p}_1 \vec{p}_2 \cdots \vec{p}_n | \vec{k}_1 \vec{k}_2 \cdots \vec{k}_n) \tag{2.150}$$

(note the bra on the RHS is *not* symmetrized) we can replace the symmetrized states in Eq. 2.149 with distinguishable states. In doing so, we obtain our final result:

$$\mathbb{1} = \int \prod_{i=1}^{n} \left[\frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_{p_i}} \right] \left[\frac{1}{n!} \mathcal{S}(\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_n) \right] |\vec{p}_1 \vec{p}_2 \cdots \vec{p}_n\rangle \langle \vec{p}_1 \vec{p}_2 \cdots \vec{p}_n| \tag{2.151}$$

When expressed in this form, the bosonic resolution of identity only differs from the distinguishable resolution of identity Eq. (2.128) in its multiplicative $\mathcal{S}/n!$ factor. As a result, it is common practice to perform derivations in quantum field theory as if all the particles involved are distinguishable (e.g. without the factor of $\mathcal{S}/n!$) and then reintroduce the $\mathcal{S}/n!$ factor as necessary in closing. This occurs frequently when considering 2-to-2 scattering in the center-of-momentum frame. Because the particles in such a process have equal-and-opposite 3-momentum (which must be nonzero in order to describe nontrivial scattering: $\vec{p}_1 \neq \vec{p}_2$), each identical incoming or outgoing pair contributes a factor of $\mathcal{S}(\vec{p}_1, \vec{p}_2)/2! = 1/2$ relative to the equivalent integral involving distinguishable particles. Formulas throughout textbooks and the literature will often come with a caveat that an additional 1/2 must be tacked on for each initial or final pair of identical bosons. This will be the case when we derive the elastic/inelastic unitarity constraints in Subsection 2.7.3.

Although we will not need it in this dissertation, for completeness let us next consider the fermionic resolution of identity. Because a coincidence of particle labels causes antisymmetrized kets to vanish, the concerns regarding the repetition factor $\mathcal{S}$ do not carry over to the fermionic case. Thus, the fermionic resolution of identity expressed in terms of canonical momentum ordering is

$$\mathbb{1} = \int_{\text{unique}} \prod_{i=1}^{n} \left[\frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_{p_i}} \right] |\vec{p}_1 \vec{p}_2 \cdots \vec{p}_n] \, [\vec{p}_1 \vec{p}_2 \cdots \vec{p}_n| \tag{2.152}$$

and generalizes to

$$\mathbb{1} = \int \prod_{i=1}^{n} \left[\frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_{p_i}} \right] \frac{1}{n!} |\vec{p}_1 \vec{p}_2 \cdots \vec{p}_n\rangle \langle \vec{p}_1 \vec{p}_2 \cdots \vec{p}_n| \tag{2.153}$$

As mentioned following the derivation of the bosonic resolution of identity, derivations in quantum field theory are often performed while assuming all particles are distinguishable and any necessary factors due to identical particles are appended after the fact. In the fermionic case, that factor is $1/n!$, which again simplifies to 1/2 for each identical fermion pair in 2-to-2 scattering processes.

2.4.3 External States: General Quantum Numbers

While the previous results were derived and motivated by considering 4-momentum eigenstates, they readily generalize to kets labeled by other sets of quantum numbers. Suppose we have a complete set of single-particle kets $|\alpha\rangle$ that resolve the single-particle identity according to

$$\mathbb{1} = \int d\Pi(\alpha) \; |\alpha\rangle\langle\alpha| \tag{2.154}$$

where $\int d\Pi(\alpha)$ is in principle some combination of sums (for discrete quantum numbers), integrals (for continuous quantum numbers), and multiplicative weights, and with normalization

$$\langle\alpha|\alpha'\rangle = w(\alpha)\,\delta_{\alpha,\alpha'} \tag{2.155}$$

where $\delta_{\alpha,\alpha'}$ is a product of Kronecker deltas (for discrete quantum numbers) and Dirac deltas (for continuous quantum numbers). Together, these imply

$$|\alpha'\rangle = \int d\Pi(\alpha)\;\, w(\alpha)\,\delta_{\alpha,\alpha'}\,|\alpha'\rangle \qquad\Longrightarrow\qquad d\Pi(\alpha) = \frac{1}{w(\alpha)}d\alpha \tag{2.156}$$

where $d\alpha$ is the differential integration element of the continuous quantum numbers specified by $|\alpha\rangle$. For example, in the previous subsection, $\alpha = \vec{p}$, such that $w(\vec{p}) = (2\pi)^3(2E_{\vec{p}})$ and $d\alpha = d^3\vec{p}$. Because kets labeled by continuous quantum numbers have Dirac delta normalizations, wavepackets corresponding to those continuous quantum numbers must be utilized in practice (refer to the discussion at the end of Subsection 2.4.1 for more details on this use of wavepackets). The construction of multi-particle states goes through without significant modification (e.g. two labels α and α' are now considered repeated if all of the quantum numbers between them are equal), such that we define the distinguishable n-particle state as

$$|\alpha_1\cdots\alpha_n\rangle = |\alpha_1\rangle\otimes\cdots\otimes|\alpha_n\rangle \tag{2.157}$$

and the identical n-particle states as

$$|\alpha_1\cdots\alpha_n) = \frac{1}{\sqrt{n!\,\mathcal{S}(\alpha_1,\cdots,\alpha_n)}}\sum_{\pi\in\pi_n}|\pi(\alpha_1,\alpha_2,\cdots,\alpha_n)\rangle \tag{2.158}$$

$$|\alpha_1\cdots\alpha_n] = \frac{1}{\sqrt{n!}}\sum_{\pi\in\pi_n}\text{sign}(\pi)\,|\pi(\alpha_1,\alpha_2,\cdots,\alpha_n)\rangle \tag{2.159}$$

for bosons and fermions respectively. In that same order, the resolutions of identity for each of these spaces equal

$$\mathbb{1} = \int\prod_{i=1}^{n}d\Pi(\alpha_i)\;\;|\alpha_1\cdots\alpha_n\rangle\langle\alpha_1\cdots\alpha_n| \tag{2.160}$$

$$\mathbb{1} = \int\prod_{i=1}^{n}d\Pi(\alpha_i)\;\frac{1}{n!}\,\mathcal{S}(\alpha_1,\cdots,\alpha_n)\,|\alpha_1\cdots\alpha_n)(\alpha_1\cdots\alpha_n| \tag{2.161}$$

$$\mathbb{1} = \int\prod_{i=1}^{n}d\Pi(\alpha_i)\;\frac{1}{n!}\,|\alpha_1\cdots\alpha_n]\![\alpha_1\cdots\alpha_n| \tag{2.162}$$

and the kets have normalizations

$$\langle\alpha_1\,\cdots\,\alpha_n|\alpha_1'\,\cdots\,\alpha_n'\rangle = \prod_{i=1}^{n} w(\alpha_i)\,\delta_{\alpha_i,\alpha_i'} \tag{2.163}$$

$$(\alpha_1\,\cdots\,\alpha_n|\alpha_1'\,\cdots\,\alpha_n') = \sum_{\text{unique }\pi\in\pi_n} \langle\pi(\alpha_1,\cdots,\alpha_n)|\alpha_1'\cdots\alpha_n'\rangle \tag{2.164}$$

$$[\alpha_1\,\cdots\,\alpha_n|\alpha_1'\,\cdots\,\alpha_n'] = \sum_{\pi\in\pi_n} \text{sign}(\pi)\,\langle\pi(\alpha_1,\cdots,\alpha_n)|\alpha_1'\cdots\alpha_n'\rangle \tag{2.165}$$

where $\mathcal{S}$ is defined as in Eq. (2.135). These general results will become relevant as we consider maximally-commuting sets of observables and thereby introduce more quantum numbers to our state labels. Note the fermionic states still obey the Pauli exclusion principle ($[\alpha\alpha\cdots| = 0$). Also note the rule of thumb that an extra factor of 1/2 should be included per identical particle pair in a 2-to-2 COM scattering calculation that was otherwise performed with distinguishable particles carries over to these more general descriptions as well.

2.4.4 S-Matrix, Matrix Element

We can use the multiparticle states defined in the previous subsection as our initial and final states in scattering processes. The collection of all states regardless of differing particle numbers and particle species content yields a Fock space, which equals the direct product of the zero-particle, single-particle, two-particle, etc. Hilbert spaces. Scattering processes are modeled as beginning in the infinite past (at time $t = -$) and ending in the infinite future (at time $t = +$) with the interesting dynamics occurring near $t = 0$. A Fock space state set up in the infinite past is called an "in state", whereas a Fock space state set up in the infinite future is an called an "out state." We can evolve an in state to out state via a generalization of the time-evolution operator $\hat{S}$ called the S-matrix:

$$\hat{S}|i\rangle_{\text{in}} = |i\rangle_{\text{out}} \tag{2.166}$$

The S-matrix $\hat{S}$ by construction (because of its relation to a time-evolution operator) commutes with P^μ and $\vec{J}$. Because our in and out states will always have definite total 4-momentum, they will also generate a 4-momentum conserving delta function, which we can preemptively factor out:

$${}_{\text{out}}\langle f|i\rangle_{\text{out}} = {}_{\text{out}}\langle f|\hat{S}|i\rangle_{\text{in}} = \mathbb{1}_{f,i} + i(2\pi)^4\delta^4(p_f - p_i)\,{}_{\text{out}}\langle f|\hat{T}|i\rangle_{\text{in}} \tag{2.167}$$

where $|i\rangle_{\text{in}}$ is some initial particle scattering state and $|f\rangle_{\text{out}}$ is some final particle scattering state. Eq. (2.167) defines the T-matrix relative to the S-matrix element, as well as the (Lorentz-invariant) matrix element

$$\mathcal{M}_{i\to f} \equiv {}_{\text{out}}\langle f|\hat{T}|i\rangle_{\text{in}} \tag{2.168}$$

The square of the matrix element is related to the probability that a given scattering process $i \to f$ will occur, and is a central topic of this dissertation.

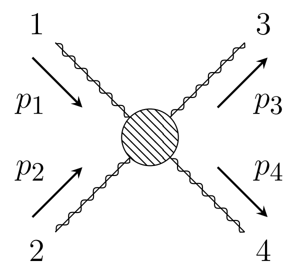
2.5 2-to-2 Scattering

This dissertation is largely concerned with 2-to-2 scattering processes, so it is important that we establish a consistent choice of conventions relating to those processes. Subsection 2.5.1 describes our parameterization of 2-to-2 scattering processes in terms of the Mandelstam variables s, t, and u. Subsection 2.5.2 defines the center-of-momentum (COM) frame and (in this frame) rewrites the aforementioned t and u Mandelstam variables in terms of s and the outgoing scattering angles θ, ϕ. Subsection 2.5.3 describes how to reduce a generic Lorentz-invariant integral over the final state particle pair degrees of freedom into a standard angular integral in the COM frame.

2.5.1 Mandelstam Variables

A 2-to-2 scattering process refers to the evolution of a two particle state in the infinite past into a two particle state in the infinite future. For the time being, we will label the particles in the incoming pair as 1 and 2, and the particles in the outgoing pair as 3 and 4. The initial and final two-particle states can be expressed with various quantum numbers in principle. For the duration of this dissertation, we will choose each external single-particle state to have definite 4-momentum p_i and helicity λ_i. The discussion of helicity is delayed until Section 2.7. By definition, an external particle with 4-momentum p_i has mass $m_i = \sqrt{p_i^2}$.

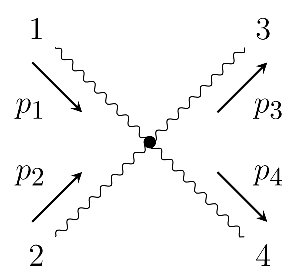
Diagrammatically, we express the aforementioned generic 2-to-2 scattering process by:



which is intended to be read from left to right, and where 4-momentum conservation guarantees

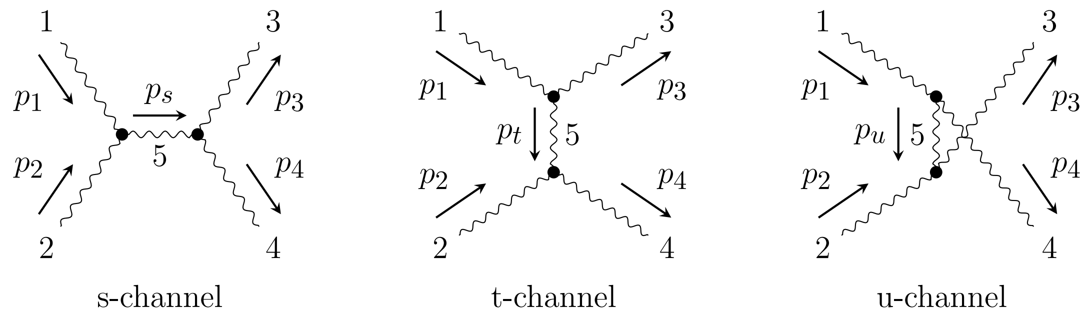
$$p_1 + p_2 = p_3 + p_4 \tag{2.169}$$

A 2-to-2 scattering process can often occur in a variety of ways via a variety of interactions. For example, depending on the details of the field theory describing this scattering process, the $(1,2)$ particles might be able to directly become a $(3,4)$ pair through a local quartic interaction. We call a diagram corresponding to this specific subprocess a contact diagram:



contact

Furthermore, if the appropriate cubic interactions are present, then this 2-to-2 scattering process is also facilitated by various channels of virtual particle exchange, i.e.



where 5 denotes the virtual particle being exchanged in each diagram. 4-momentum is conserved at each vertex, such that $p_s = p_1 + p_2$, and $p_1 = p_t + p_3$, and so-on. These diagrams are the motivation for the Mandelstam variables [22], which are defined as follows:

$$s \equiv p_s^2 = (p_1 + p_2)^2 = (p_3 + p_4)^2 \tag{2.170}$$
$$t \equiv p_t^2 = (p_1 - p_3)^2 = (p_4 - p_2)^2 \tag{2.171}$$
$$u \equiv p_u^2 = (p_1 - p_4)^2 = (p_3 - p_2)^2 \tag{2.172}$$

Note that s (t; u) is the invariant momentum-squared that flows through the virtual particle in an s-channel (t-channel; u-channel) exchange diagram. Although the Mandelstam variables are motivated by these exchange diagrams, we may express any 2-to-2 scattering process in terms of s, t, and u. Indeed, we will be using s as a convenient variable to track energy growth for all kinds of diagrams.

Mandelstam s, t, and u are not independent variables. For example, their sum is constrained: through direct evaluation, we find

$$s + t + u = (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2 \tag{2.173}$$
$$= p_1^2 + p_2^2 + p_3^2 + p_4^2 + \underbrace{2p_1 \cdot (p_1 + p_2 - p_3 - p_4)}_{=0 \text{ by 4-momentum conservation}} \tag{2.174}$$

such that

$$s + t + u = \sum_{i=1}^{4} m_i^2 \tag{2.175}$$

Furthermore, the Mandelstam variables are real-valued with restricted range when describing experimentally-allowed processes. Mandelstam s, for example, is never smaller than

$$s_{\text{min}} \equiv \max\left[(m_1 + m_2)^2, (m_3 + m_4)^2\right] \tag{2.176}$$

which corresponds to both particles of either the initial or final particle pair being at rest, based on which pair is more massive overall (because of 4-momentum conservation, heavier particles at rest can become lighter particles in motion, but not vice-versa). Consequently, Mandelstam s only vanishes when all external particles are massless and the 3-momenta between the particles in each pair are collinear. Because collinear massless wavepackets will never collide, s will never vanish for nontrivial scattering processes.

Until now, our discussion has been frame independent. Let us now consider a special frame that is particularly useful for simplifying scattering calculations: the center-of-momentum frame.

2.5.2 Center-Of-Momentum Frame

As remarked in the previous subsection, $s = (p_1 + p_2)^2$ is nonzero for any nontrivial 2-to-2 scattering process. Like a massive single-particle state with positive squared 4-momentum, such a process possesses a (2-particle) rest frame, wherein the particle pair's total 3-momentum vanishes: $\vec{p}_1 + \vec{p}_2 = \vec{0}$. This property (in addition to some coordinate decisions we detail shortly) defines the center-of-momentum (COM) frame. So long as $s > 0$, we may always use some combinations of boosts and rotations to enter the COM frame. For example, we only need an appropriately-chosen boost to ensure the total 3-momentum of the system vanishes, or in other words that the incoming particles have equal-and-opposite 3-momenta:

$$\vec{p}_1 + \vec{p}_2 = \vec{0} \tag{2.177}$$

which (via 4-momentum conservation) implies the outgoing particles have equal-and-opposite 3-momenta as well:

$$\vec{p}_3 + \vec{p}_4 = \vec{0} \tag{2.178}$$

Geometrically, this means that in the COM frame the 3-momentum of the incoming particle pair lie on one common line and the 3-momentum of the final particle pair lie on another. Furthermore, this boost uniquely determines the 3-momentum magnitudes of the external particles: namely,

$$|\vec{p}_1| = |\vec{p}_2| = \mathbb{P}(1,2) \qquad\qquad |\vec{p}_3| = |\vec{p}_4| = \mathbb{P}(3,4) \tag{2.179}$$

where

$$\mathbb{P}(i,j) = \sqrt{\frac{1}{4s}\left[s - (m_i - m_j)^2\right]\left[s - (m_i + m_j)^2\right]} \tag{2.180}$$

Next, we can use a rotation to orient the 3-momentum of particle 1 in the $\hat{z}$ direction (or, equivalently, we can define the $\hat{z}$ direction of our coordinate system such that it follows $\vec{p}_1$ so long as $|\vec{p}_1|$ is nonzero), such that

$$p_1 = E_1\hat{t} + |\vec{p}|_1\,\hat{z} \tag{2.181}$$

$$p_2 = E_2\hat{t} - |\vec{p}|_1\,\hat{z} \tag{2.182}$$

and

$$p_3 = E_3\hat{t} + |\vec{p}|_3\,\hat{p}_3 \tag{2.183}$$

$$p_4 = E_4\hat{t} - |\vec{p}|_3\,\hat{p}_3 \tag{2.184}$$

where the basis 4-vectors were defined at the end of Section (2.2.1). This completes our definition of the COM frame. We choose to express $\hat{p}_3$ in spherical coordinates with respect to $\hat{z}$ in the usual way, such that $\hat{p}_3^\mu = (0, c_\theta s_\phi, s_\theta s_\phi, c_\phi)$. We remind the reader that all of the external energies are restricted by the on-shell condition $m_i^2 = p_i^2 = E_i^2 - |\vec{p}_i|^2$, such that (via Eq. (2.179)) all external 4-momenta can be expressed in terms of the s, θ, ϕ, and the particle masses.

Because the 3-momenta of the incoming particles 1 and 2 are equal-and-opposite in the COM frame, Mandelstam s reduces to the square of the total incoming energy, which we denote E_{COM}:

$$s = (p_1 + p_2)^2 = (E_1 + E_2)^2 \equiv E_{\text{COM}}^2 \tag{2.185}$$

When context makes ambiguity unlikely (i.e. it is apparent that we are not referring to a single-particle energy), we will drop the label from E_{COM} and simply write $s = E^2$.

Like the external 4-momenta, we can express the Mandelstam variables t and u in terms of s, θ, and ϕ. To do so with succinctness, it is useful to define

$$\mathbb{P}(i,j,k,l) = \sqrt{\frac{1}{4s}\left[s^2 - (m_k^2 + m_l^2 + m_m^2 + m_n^2)s + (m_k^2 - m_l^2)(m_m^2 - m_n^2)\right]} \tag{2.186}$$

where the previously-defined $\mathbb{P}(i,j)$ equals $\mathbb{P}(i,j,i,j)$. Then the Mandelstam variables equal

$$t(s,\theta) = 2\Big[-\mathbb{P}(1,2,3,4)^2 + \cos(\theta)\,\mathbb{P}(1,2)\cdot\mathbb{P}(3,4)\Big] \tag{2.187}$$

$$u(s,\theta) = 2\Big[-\mathbb{P}(1,2,4,3)^2 - \cos(\theta)\,\mathbb{P}(1,2)\cdot\mathbb{P}(3,4)\Big] \tag{2.188}$$

Note these are all independent of ϕ, which cancels out despite its presence in p_3 and p_4.

For future use in elastic processes, it is useful to define one last simplification of $\mathbb{P}(i,j,k,l)$:

$$\mathbb{P}(i) = \mathbb{P}(i,i,i,i) = \frac{1}{2}\sqrt{s - 4m_i^2} \tag{2.189}$$

For example, in elastic scattering (where all external particles are of identical particle species, say, 1),

$$t(s,\theta)|_{\text{elastic}} = 2\mathbb{P}(1)^2\Big[-1 + \cos(\theta)\Big] = -\frac{1}{2}(s - 4m_1^2)[1 - \cos(\theta)] \tag{2.190}$$

$$u(s,\theta)|_{\text{elastic}} = 2\mathbb{P}(1)^2\Big[-1 - \cos(\theta)\Big] = -\frac{1}{2}(s - 4m_1^2)[1 + \cos(\theta)] \tag{2.191}$$

Before discussing the quantum theory of 2-to-2 scattering, there is one more result we require. This subsection demonstrated that once an incoming energy $E_{\text{COM}} = \sqrt{s}$ is set, the only remaining degrees of freedom (ignoring internal degrees of freedom like helicity) correspond to the outgoing angles θ and ϕ. To derive the optical theorem (in Subsection 2.5.4) in a form that then allows us to derive the partial wave elastic/inelastic unitarity constraints (in Subsection 2.7.3), we would like to rewrite a 2-particle Lorentz invariant integral in terms of the remaining variables θ and ϕ. This is the subject of the next subsection.

2.5.3 2-Particle Lorentz Invariant Integrals in the COM Frame

There are several occasions when an integral over a final state particle pair is necessary. For example, such an integral is required when we calculate the total cross-section for a given 2-to-2 scattering process and are uninterested in the specific outgoing angle of the final pair. This kind of integral also occurs when deriving the partial wave elastic/inelastic unitarity constraints, which are important for this dissertation.

For the 2-to-2 scattering process $1\,2 \to 3\,4$, an outgoing particle pair integral is typically written as

$$\mathfrak{F} \equiv \int \underbrace{\left[\frac{d^3p_3}{(2\pi)^3}\frac{1}{2E_3}\right]\left[\frac{d^3p_4}{(2\pi)^3}\frac{1}{2E_4}\right]}_{\text{2-Particle Lorentz-Invariant Phase Space}} \underbrace{\left[(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\right]}_{\text{4-Momentum Conservation}} F(p_3,p_4) \tag{2.192}$$

independent of frame, where F is a generic function of the final particle 4-momenta. We aim to use the four Dirac deltas present to eliminate four of the six integration parameters and thereby rewrite $\mathfrak{F}$ as a two-dimensional integral. In particular, we perform this integral in the COM frame, and so the goal is to have those final two integration parameters be θ and ϕ, which describe the direction of $\hat{p}_3$ relative to $\hat{p}_1 = \hat{z}$.

In the COM frame, $p_1 = (E_1, \vec{p}_1)$ and $p_2 = (E_2, -\vec{p}_1)$, and the Dirac delta becomes

$$\delta^4(p_1+p_2-p_3-p_4) = \delta(E_{\text{COM}} - E_3 - E_4)\,\delta^3(\vec{p}_3+\vec{p}_4) \tag{2.193}$$

where $E_{\text{COM}} = E_1 + E_2$. The 3-vector Dirac delta $\delta^3(\vec{p}_3+\vec{p}_4)$ allows us to immediately eliminate the d^3p_4 integral by constraining $\vec{p}_4 = -\vec{p}_3$, such that we may write

$$\mathfrak{F} = \frac{1}{16\pi^2}\int \frac{d^3p_3}{E_3E_4}\,\delta(E_{\text{COM}} - E_3 - E_4)\,F(p_3,p_4)\bigg|_{\vec{p}_4=-\vec{p}_3} \tag{2.194}$$

Meanwhile, the integration measure d^3p_3 is expressible in spherical coordinates like so

$$d^3p_3 = |\vec{p}_3|^2\,d|\vec{p}_3|\,d\Omega = \frac{1}{2}\,|\vec{p}_3|\,d|\vec{p}_3|^2\,d\Omega \tag{2.195}$$

where $d\Omega = d\cos\theta\,d\phi$ contains the integration variables we wish to retain. Therefore, we want to use the final Dirac delta $\delta(E_{\text{COM}} - E_3 - E_4)$ remaining in $\mathfrak{F}$ to eliminate the $d|\vec{p}_3|^2$ integral. To do so, we must reparameterize the Dirac delta using the following property:

$$\delta(f(x)) = \sum_{x_* \text{ s.t. } f(x_*)=0} \frac{\delta(x-x_*)}{|f'(x_*)|} \tag{2.196}$$

As mentioned in the previous section, 4-momentum conservation is satisfied (and thus $E_{\text{COM}} = E_3 + E_4$) precisely when $|\vec{p}_3| = \mathbb{P}(3,4)$. Furthermore, using the existing $\vec{p}_4 = -\vec{p}_3$

constraint,

$$\frac{\partial}{\partial|\vec{p}_3|^2}\Big[E_{\rm COM}-E_3-E_4\Big]=\frac{\partial}{\partial|\vec{p}_3|^2}\Big[E_{\rm COM}-\sqrt{m_3^2+|\vec{p}_3|^2}-\sqrt{m_4^2+|\vec{p}_3|^2}\Big] \tag{2.197}$$

$$=-\frac{1}{2}\left[\frac{1}{\sqrt{m_3^2+|\vec{p}_3|^2}}+\frac{1}{\sqrt{m_4^2+|\vec{p}_3|^2}}\right] \tag{2.198}$$

$$=-\frac{1}{2}\frac{E_3+E_4}{E_3E_4} \tag{2.199}$$

Hence, utilizing the fact that the Dirac delta vanishes whenever $E_{\rm COM}\neq E_3+E_4$,

$$\delta(E_{\rm COM}-E_3-E_4)=\frac{2E_3E_4}{E_{\rm COM}}\,\delta\Big(|\vec{p}_3|^2-\mathbb{P}(3,4)^2\Big) \tag{2.200}$$

and, thus,

$$\mathfrak{F}=\frac{\mathbb{P}(3,4)}{16\pi^2E_{\rm COM}}\int d\Omega\quad F(p_3,p_4)\Big|_{\vec{p}_3=\mathbb{P}(3,4)\hat{p}_3=-\vec{p}_4} \tag{2.201}$$

where $\mathbb{P}(3,4)$ is defined in Eq. 2.180. This is the desired result.

2.5.4 The Optical Theorem

The S-matrix (defined in Subsection 2.4.4) is a unitary operator on Fock space that encodes how initial particle configurations evolve into final state particle configurations. Because it is unitary, S-matrix elements must satisfy

$$\mathbb{1}_{\bar{i},i}={}_{\rm in}\langle\bar{i}|\hat{S}^\dagger\hat{S}|i\rangle_{\rm in}=\sum_f\int d\Pi(f)\quad {}_{\rm in}\langle\bar{i}|\hat{S}^\dagger|f\rangle_{\rm out}\;\;{}_{\rm out}\langle f|\hat{S}|i\rangle_{\rm in} \tag{2.202}$$

$$=\sum_f\int d\Pi(f)\quad {}_{\rm out}\langle f|\hat{S}|\bar{i}\rangle^*_{\rm in}\;\;{}_{\rm out}\langle f|\hat{S}|i\rangle_{\rm in} \tag{2.203}$$

where we have inserted the Fock space resolution of identity and embedded the necessary state normalization weights into $d\Pi(f)$. We would like to recast this constraint in terms of the corresponding matrix elements $\mathcal{M}_{i\to f}$. To do so, suppose $p_i=p_{\bar{i}}$, and note

$${}_{\rm out}\langle f|\hat{S}|\bar{i}\rangle^*_{\rm in}\;\;{}_{\rm out}\langle f|\hat{S}|i\rangle_{\rm in}=\Big[\mathbb{1}_{\bar{i},f}-i(2\pi)^4\delta^4(p_{\bar{i}}-p_f)\,\mathcal{M}^*_{\bar{i}\to f}\Big] \tag{2.204}$$

$$\Big[\mathbb{1}_{f,i}+i(2\pi)^4\delta^4(p_f-p_i)\,\mathcal{M}_{i\to f}\Big] \tag{2.205}$$

$$=\mathbb{1}_{\bar{i},f}\mathbb{1}_{f,i}+i(2\pi)^4\delta^4(p_f-p_i)\Big[\mathcal{M}_{i\to f}\mathbb{1}_{\bar{i},f}-\mathcal{M}^*_{\bar{i}\to f}\mathbb{1}_{i,f}\Big] \tag{2.206}$$

$$+\Big[(2\pi)^4\delta^4(p_i-p_f)\Big]^2\mathcal{M}^*_{\bar{i}\to f}\mathcal{M}_{i\to f} \tag{2.207}$$

The squared Dirac delta in the final term can be understood by considering a finite volume universe wherein the Dirac delta is replaced with a Kronecker delta; however, we simply use this expression as written in the RHS of Eq. (2.203), and eliminate one Dirac delta from the pair via $\sum_f \int d\Pi(f)$. (If we had not assumed $p_i = p_{\bar{i}}$ before now, the Dirac delta pair would have enforced their equality for this term.) In entirety, this substitution yields

$$-i\left[\mathcal{M}_{i\to\bar{i}} - \mathcal{M}^*_{\bar{i}\to i}\right] = \sum_f \int d\Pi(f) \quad (2\pi)^4\delta^4(p_i - p_f)\mathcal{M}^*_{\bar{i}\to f}\mathcal{M}_{i\to f} \tag{2.208}$$

In particular, if $\bar{i} = i$ (and not just $p_i = p_{\bar{i}}$ as previously assumed), then

$$2\Im[\mathcal{M}_{i\to i}] = \sum_f \int d\Pi(f) \quad (2\pi)^4\delta^4(p_i - p_f)|\mathcal{M}_{i\to f}|^2 \tag{2.209}$$

where $\Im$ denotes the imaginary part of its argument ($\Re$ similarly denotes a real part). Eq. (2.209) is the optical theorem.

We are interested in applying the optical theorem to 2-to-2 scattering processes in the COM frame. To facilitate this application, first divide the sum over processes on the RHS of Eq. (2.209) into two groups: n-to-two scattering ($f = f_2$) processes, and the rest. This yields two sums

$$\sum_{f_2} \int d\Pi(f_2) \quad (2\pi)^4\delta^4(p_{f_2} - p_i)|\mathcal{M}_{i\to f_2}|^2 + \underbrace{\sum_{f\neq f_2} \int d\Pi(f) \quad (2\pi)^4\delta^4(p_i - p_f)|\mathcal{M}_{i\to f}|^2}_{\equiv C_{f\neq f_2}\geq 0} \tag{2.210}$$

If we assume our external states have well-defined 4-momentum quantum numbers, then (aside from potential sums and integrals over additional quantum numbers) the first term contains an integral precisely of the form we simplified in the previous subsection. Therefore, we can rewrite it as

$$\int \Pi(f_2) \quad (2\pi)^4\delta^4(p_i - p_3 - p_4)f(\theta,\phi) = \frac{\mathbb{P}(3,4)}{16\pi^2 E_i}\int d\Pi(f_2^*) \int d\Omega \quad f(\theta,\phi) \tag{2.211}$$

where $d\Pi(f_2^*)$ includes sums or integrals over any other relevant quantum numbers beyond 4-momenta. Substituting this into Eq. (2.209), the optical theorem now yields

$$2\Im[\mathcal{M}_{i\to i}] = \sum_{f_2}\frac{\mathbb{P}(3,4)}{16\pi^2 E_i}\int d\Omega \quad |\mathcal{M}_{i\to f_2}|^2 + C_{f\neq f_2} \tag{2.212}$$

We will further reduce this in Section 2.7 with the help of the partial wave amplitude decomposition. However, before we define the partial wave decomposition of a matrix element, we first recount the rotational machinery, notation, and conventions of quantum mechanics which the decomposition relies on.

2.6 Angular Momentum

As remarked in Subsection 2.3.2, angular momentum operators $\vec{J}$ generate representations of the Lie group $\mathbf{SU(2)}$ despite being associated with representations of $\mathbf{SO(3)}$ prior to quantum promotion. This section reviews the derivation of all irreducible finite-dimensional unitary representations of $\mathbf{SU(2)}$, the combination of $\mathbf{SU(2)}$ representations via the Clebsh-Gordan machinery, and the Wigner D matrix. Because this topic is standard in quantum mechanics texts, we outline results for the sake of reference (and establishing convention) rather than pedagogy.

2.6.1 Finite-Dimensional Angular Momentum Representations

The angular momentum operators satisfy the $\mathbf{SU}(2)$ commutation relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k \qquad \Longrightarrow \qquad \vec{J}\times\vec{J} = i\vec{J} \tag{2.213}$$

which we obtain from the 4-vector equivalent Eq. (2.40) by replacing $J_i \mapsto -iJ_i$ according to the procedure described in Subsection 2.3.1. As before, every angular momentum operator commute with the total angular momentum operator $\vec{J}^2$, which is the only Casimir operator of $\mathbf{SU(2)}$: for example,

$$\begin{aligned}
[J_z, \vec{J}^2] &= \sum_{j=1}^{3}[J_z, J_j]J_j + J_j[J_z, J_j] &(2.214)\\
&= [J_z, J_x]J_x + [J_z, J_y]J_y + J_x[J_z, J_x] + J_y[J_z, J_y] &(2.215)\\
&= iJ_yJ_x - iJ_xJ_y + iJ_xJ_y - iJ_yJ_x &(2.216)\\
&= 0 &(2.217)
\end{aligned}$$

which, by cyclic symmetry, means

$$[\vec{J}, \vec{J}^2] = \vec{0} \tag{2.218}$$

As is standard, we choose our maximally-commuting set of observables in $\mathbf{SU(2)}$ to be $\{J^2, \vec{J}_z\}$, such that our kets satisfy

$$\vec{J}^2|j,m\rangle = c_j|j,m\rangle \qquad J_z|j,m\rangle = m|j,m\rangle \tag{2.219}$$

for a soon-to-be-determined real number c_j. We also choose to normalize these states such that

$$\langle j,m|j',m'\rangle = \delta_{jj'}\delta_{mm'} \tag{2.220}$$

It is in this basis that we begin the process of deriving all irreducible finite-dimensional representations. Just as we were able to relate kets with different 4-momentum on the same mass hyperboloid using Lorentz transformations, we can relate different eigenstates of J_z having the same eigenvalue of $\vec{J}^2$ via the ladder operators

$$J_\pm = J_x \pm iJ_y \tag{2.221}$$

The ladder operators cannot change the eigenvalue of $\vec{J}^2$ because $\vec{J}^2$ commutes with every angular momentum operator and thus $J_\pm$ as well. Note that $J_\pm^\dagger = J_\mp$. Also note that

$$J_\pm J_\mp = (J_x \pm iJ_y)(J_x \mp iJ_y) = J_x^2 + J_y^2 \mp i[J_x, J_y] = \vec{J}^2 - J_z^2 \mp J_z \tag{2.222}$$

such that

$$\vec{J}^2 = J_\pm J_\mp + J_z^2 \pm J_z \tag{2.223}$$

and

$$[J_z, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_z \tag{2.224}$$

These allow us to confirms that the ladder operators do in fact change the eigenvalue of J_z in a well-defined way:

$$J_z J_\pm |jm\rangle = \Big[J_\pm J_z + [J_z, J_\pm] \Big] |j,m\rangle \tag{2.225}$$

$$= \Big[J_\pm J_z \pm J_\pm \Big] |j,m\rangle \tag{2.226}$$

$$= (m \pm 1) J_\pm |j,m\rangle \tag{2.227}$$

or, in other words,

$$J_\pm |j,m\rangle \propto |j, m \pm 1\rangle \tag{2.228}$$

up to some overall phase and normalization. Therefore, by repeatedly applying instances of J_+ and J_- to a ket $|m\rangle$, we can seemingly construct a ket $|m+n\rangle$ with J_z eigenvalue $m+n$ for any integer n. However, we desire a finite-dimensional representation, so there must exist some real number $m_{\text{max}} \equiv j \equiv m+n$ such that its eigenvalue cannot be raised any further, e.g. $J_+|j,j\rangle = 0$. For this state,

$$\vec{J}^2 |j,j\rangle = \Big[J_- J_+ + J_z^2 + J_z \Big] |j,j\rangle = j(j+1)|j,j\rangle \tag{2.229}$$

Thus, for this maximal J_z state with J_z eigenvalue j, it has definite $\vec{J}^2$ eigenvalue $j(j+1)$. Because $[J_z, \vec{J}^2] = 0$, all J_z eigenkets that are related to each other by ladder operators have the same $\vec{J}^2$ eigenvalue. Hence, the earlier c_j equals $j(j+1)$, such that

$$J_z|j,m\rangle = m|j,m\rangle \qquad \vec{J}^2|j,m\rangle = j(j+1)|j,m\rangle \tag{2.230}$$

By combining $J_z J_\pm |j,m\rangle = (m \pm 1) J_\pm |j,m\rangle$ from Eq. (2.227) and

$$\langle jm| J_\pm^\dagger J_\pm |j,m\rangle = \langle jm| J_\mp J_\pm |j,m\rangle \tag{2.231}$$

$$= \langle jm| \Big[\vec{J}^2 - J_z^2 \pm J_z \Big] |j,m\rangle \tag{2.232}$$

$$= \Big[j(j+1) - m^2 \pm m \Big] \delta_{jj'} \delta_{mm'} \tag{2.233}$$

we find, noting $j(j+1) - m^2 \pm m = (j \mp m)(j \pm m + 1)$ as to rewrite the denominator factor into a standard form,

$$|j, m \pm 1\rangle = \frac{J_\pm}{\sqrt{(j \mp m)(j \pm m + 1)}} |j, m\rangle \tag{2.234}$$

where an undetermined phase has been set to 1 via the Condon-Shortley phase convention.

Note that the demand for a finite dimensional representation works on both extremes of the J_z eigenvalue spectrum: instead of demanding $J_+|j, m\rangle$ vanish for some value of $m = j \equiv m_{\text{max}}$ (i.e. the J_z eigenvalue can be raised no further), we can seek the value $m = m_{\text{min}}$ such that $J_-|j, m\rangle$ vanishes (i.e. the J_z eigenvalue can be lowered no further). This for this value, we find

$$j(j+1)|j, m_{\text{min}}\rangle = \vec{J}^2|j, m_{\text{min}}\rangle = \left[J_+J_- + J_z^2 - J_z\right]|j, m_{\text{min}}\rangle = m_{\text{min}}(m_{\text{min}} - 1)|j, m_{\text{min}}\rangle \tag{2.235}$$

which implies m_{min} must equal either $-j$ or $j+1$. By definition m_{min} cannot exceed m_{max}, so it must be the case that $m_{\text{min}} = -j$. Finally, because the ladder operators only change J_z eigenvalues by integer amounts, the range of the spectrum $j - (-j) = 2j$ must be an integer as well, and thus j must be a half-integer. With this, our construction of the representation is complete.

To summarize: there exists a $(2j+1)$-dimensional representation of $\mathbf{SU(2)}$ for every nonnegative half-integer j composed of kets $|jm\rangle$ that satisfy $\vec{J}^2|jm\rangle = j(j+1)|jm\rangle$ and $J_z|jm\rangle = m|jm\rangle$, where the integer m ranges from $-j$ to j. We choose our normalizations and phases for these states as follows:

$$\langle jm|j'm'\rangle = \delta_{jj'}\,\delta_{mm'} \tag{2.236}$$

such that

$$\mathbb{1} = \sum_{j=0}^{+\infty} \sum_{m=-j}^{+j} |j, m\rangle\langle j, m| \tag{2.237}$$

and

$$|j, m \pm 1\rangle = \frac{J_\pm}{\sqrt{(j \mp m)(j \pm m + 1)}} |j, m\rangle \tag{2.238}$$

where $J_\pm = J_x \pm iJ_y$.

For a spin-j massless particle, a similar construction will be useful for describing their helicity eigenstates, which have two possible values: $\lambda = \pm j$. However, because massless particles lack longitudinal helicity modes, we generally cannot relate the $\lambda = +j$ and $\lambda = -j$ helicity states via the ladder operators. Instead, we relate them via the reflection operator

$$Y \equiv \mathcal{U}[R_y(\pi)]\,\mathcal{U}[P] \tag{2.239}$$

where $\mathcal{U}[P]$ is the quantum equivalent of the parity operator P [23]. Because the angular momentum generators commute with the parity operator ($[J^i, P] = 0$), the angular momentum eigenstates are at most changed by a phase

$$\mathcal{U}[P]|j, m\rangle \propto |j, m\rangle \tag{2.240}$$

whereas, from the angular momentum commutation relations and exisiting definitions,

$$\mathcal{U}[R_y(\pi)]|j, m\rangle = e^{-i\pi J_y}|j, m\rangle = (-1)^{j-m}|j, -m\rangle \tag{2.241}$$

We choose these phases such that

$$Y\,|j, m\rangle = \eta\,|j, -m\rangle \tag{2.242}$$

for an undetermined phase η called the parity factor of the corresponding particle species. Note that when acted on a 4-momentum p, the equivalent 4-vector representation of Y yields $Y^\mu{}_\nu p^\nu = R_y(\pi)^\mu{}_\nu\,(E, -\vec{p})^\nu = (E, p_x, -p_y, p_z)$, such that Y leaves (for example) p_z invariant.

2.6.2 Adding Angular Momentum Representations

Angular momentum eigenstates can be combined via direct product in the usual way to form a state $|j_1, m_1, j_2, m_2\rangle$ defined as

$$|j_1, m_1, j_2, m_2\rangle \equiv |j_1, m_1\rangle \otimes |j_2, m_2\rangle \tag{2.243}$$

with eigenvalue content

$$\vec{J}_1^2\,|j_1, m_1, j_2, m_2\rangle = j_1(j_1+1)\,|j_1, m_1, j_2, m_2\rangle \tag{2.244}$$

$$(J_1)_z\,|j_1, m_1, j_2, m_2\rangle = m_1\,|j_1, m_1, j_2, m_2\rangle \tag{2.245}$$

$$\vec{J}_2^2\,|j_1, m_1, j_2, m_2\rangle = j_2(j_2+1)\,|j_1, m_1, j_2, m_2\rangle \tag{2.246}$$

$$(J_2)_z\,|j_1, m_1, j_2, m_2\rangle = m_2\,|j_1, m_1, j_2, m_2\rangle \tag{2.247}$$

However, there is another basis for these two-particle states which is sometimes more useful. Define the 2-particle total angular momentum operator as

$$\vec{J} = \vec{J}_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \vec{J}_2 \tag{2.248}$$

wherein $\mathbb{1}_1$ and $\mathbb{1}_2$ are the identity operators on the first and second particle Hilbert spaces respectively. Usually the identity operators are understood from context, and we simply write $\vec{J} = \vec{J}_1 + \vec{J}_2$. Because $[(\vec{J}_1)^i, (\vec{J}_2)^j] = 0$,

$$[J^i, J^j] = [(J_1)^i, (J_1)^j] + [(J_2)^i, (J_2)^j] = \epsilon_{ijk}\left[(J_1)^k + (J_2)^k\right] = \epsilon_{ijk}J^k \tag{2.249}$$

such that $\vec{J}$ acts like the usual total angular momentum operator. Furthermore, $[\vec{J}^2, \vec{J}_1^2] = [\vec{J}^2, \vec{J}_2^2] = 0$, and so we can choose $\{\vec{J}_1^2, \vec{J}_2^2, \vec{J}^2, J_z\}$ as a maximally-commuting set of observables for a basis of states $|j_1, j_2, J, M\rangle$ with eigenvalue content

$$\vec{J}_1^2\,|j_1, j_2, J, M\rangle = j_1(j_1+1)\,|j_1, j_2, J, M\rangle \tag{2.250}$$
$$\vec{J}_2^2\,|j_1, j_2, J, M\rangle = j_2(j_2+1)\,|j_1, j_2, J, M\rangle \tag{2.251}$$
$$\vec{J}^2\,|j_1, j_2, J, M\rangle = J(J+1)\,|j_1, j_2, J, M\rangle \tag{2.252}$$
$$\vec{J}_z^2\,|j_1, j_2, J, M\rangle = M\,|j_1, j_2, J, M\rangle \tag{2.253}$$

Given eigenvalues j_1 and j_2, the $\vec{J}^2$ eigenvalue only exists for $J \in \{|j_1 - j_2|, \ldots, j_1 + j_2\}$.

We can convert between the representations using

$$|j_1, j_2, J, M\rangle = \sum_{m_1=-j_1}^{+j_1} \sum_{m_2=-j_2}^{+j_2} |j_1, m_1, j_2, m_2\rangle\langle j_1, m_1, j_2, m_2|j_1, j_2, J, M\rangle \tag{2.254}$$

where $\langle j_1, m_1, j_2, m_2|j_1, j_2, J, M\rangle$ is called a Clebsch-Gordan (CG) coefficient. People do not typically calculate Clebsch-Gordan coefficients themselves, instead using existing resources [9]. The particular CG coefficients we require at present are used to combine two $j_1 = j_2 = 1$ representations into a $J = 2$ representation. Explicitly, we obtain

$$\begin{aligned}
|2, \pm 2\rangle &= |1, \pm 1\rangle \otimes |1, \pm 1\rangle \\
|2, \pm 1\rangle &= \frac{1}{\sqrt{2}}\Big[|1, \pm 1\rangle \otimes |1, 0\rangle + |1, 0\rangle \otimes |1, \pm 1\rangle \\
|2, 0\rangle &= \frac{1}{\sqrt{6}}\Big[|1, \pm 1\rangle \otimes |1, \mp 1\rangle + |1, \mp 1\rangle \otimes |1, \pm 1\rangle + 2\,|1, 0\rangle \otimes |1, 0\rangle
\end{aligned} \tag{2.255}$$

where we suppress the $j_1 = j_2 = 1$ labels of the $|j_1, j_2, J, M\rangle$ kets on the LHS.

2.6.3 Wigner D-Matrix

The quantum equivalent of rotations are unitary operators. In particular, the generic rotation expressed in terms of Euler angles becomes

$$\mathcal{U}[R(\phi, \theta, \psi)] \equiv \mathcal{U}[R_z(\phi)]\,\mathcal{U}[R_y(\theta)]\,\mathcal{U}[R_z(\psi)] \tag{2.256}$$

where

$$\mathcal{U}[R_i(\alpha)] = \text{Exp}[-i\alpha J_i] \tag{2.257}$$

for $i \in \{x, y, z\}$, and $\vec{J}$ are the angular momentum operators. We previously defined a restricted version of the Euler angle decomposition $\mathcal{U}[R(\hat{p})] = \mathcal{U}[R(\phi, \theta)] = \mathcal{U}[R(\phi, \theta, -\phi)]$ which is sufficient for mapping a 3-momentum $|\vec{p}|\hat{z}$ to a 3-momentum $\vec{p}$. The inverse of $\mathcal{U}[R(\phi, \theta, \psi)]$ is $\tilde{\mathcal{U}}[R(\phi, \theta, \psi)] = \mathcal{U}[R(-\psi, -\theta, -\phi)]$.

Keeping in mind that the rotation operator (being a function of the angular momentum operators alone) cannot influence the eigenvalue of $\vec{J}^2$, the Wigner D-matrix $\mathcal{D}^{j}_{m_f, m_i}$ is defined as follows:

$$\mathcal{D}^{j_i}_{m_f, m_i}(\phi, \theta, \psi)\,\delta_{j_f, j_i} \equiv \langle j_f, m_f|\mathcal{U}[R(\phi, \theta, \psi)]|j_i, m_i\rangle \tag{2.258}$$

Note the Kronecker delta δ_{j_f,j_i} on the LHS. Because the Euler angles provide a natural coordinate system for a symmetric top, the Wigner D-matrix is sometimes referred to as the wavefunction of a symmetric top. The z-axis rotations in the Euler angle rotation operator Eq. (2.256) can be simplified because $J_z|j,m\rangle = m|j,m\rangle$, and doing so allows us to define the Wigner (small) d-matrix $d^j_{m_f,m_i}$ in terms of the Wigner D-matrix:

$$\mathcal{D}^{j_i}_{m_f,m_i}(\phi,\theta,\psi)\,\delta_{j_f,j_i} = e^{-i(m_f\phi+m_i\psi)}\langle j_f,m_f|\mathcal{U}[R_y(\theta)]|j_i,m_i\rangle \tag{2.259}$$

$$\equiv e^{-i(m_f\phi+m_i\psi)}\,d^{j_i}_{m_f,m_i}(\theta) \tag{2.260}$$

In particular, in terms of the restricted Euler angle decomposition (for which $\psi = -\phi$ in our convention),

$$\mathcal{D}^{j}_{m_f,m_i}(\phi,\theta) = \langle j,m_f|\mathcal{U}[R(\phi,\theta,\psi)]|j,m_i\rangle = e^{i(m_f-m_i)\phi}\,d^{j}_{m_f,m_i}(\theta) \tag{2.261}$$

where we have set $j = j_i = j_f$. The Wigner D-matrix satisfies several convenient properties. For example, if $\theta = 0$, then $\mathcal{U}[R(\phi,\theta)] = \mathbb{1}$, and

$$\mathcal{D}^{j}_{m_f m_i}(\phi,0) = \delta_{m_f,m_i} \tag{2.262}$$

Furthermore, the restricted form has a convenient orthogonality relation:

$$\int d\Omega\;\mathcal{D}^{j_1*}_{m_1\lambda}(\hat{p})\,\mathcal{D}^{j_2*}_{m_2\lambda}(\hat{p}) = \frac{4\pi}{2j+1}\,\delta_{j_1,j_2}\,\delta_{m_1,m_2} \tag{2.263}$$

The Wigner D-matrix is an important element of relativistic scattering calculations involving helicity eigenstates, which we are now prepared to address.

2.7 Helicity

2.7.1 Single-Particle States

In Subsection 2.3.4, we refined our focus to eigenstates of the Hamiltonian H with definite mass M and spin s, and thereby defined the helicity operator Λ as

$$\Lambda \equiv \frac{\vec{J}\cdot\vec{P}}{\sqrt{E^2 - M^2}} \tag{2.264}$$

on those states. As demonstrated then, Λ commutes with P^i, $\vec{P}^2$, J^i, $\vec{J}^2$, and P^2. This yields (among others) two maximally-commuting sets of observable operators, both of which involve the helicity operator:

- **Option 1:** P^μ, Λ
- **Option 2:** H, $\vec{J}^2$, J_z, Λ

in addition to the P^2 and the internal spin/helicity, the Poincaré Casimir operators. The first option will describe our external one-particle states. However, the second option allows us utilize symmetries of the S-matrix in order to derive the partial wave unitarity constraints. This section investigates the relationship between these two options.

Suppose we utilize Option 1, so that our one-particle states $|p,\lambda\rangle$ satisfy

$$H\,|p,\lambda\rangle = E\,|p,\lambda\rangle \qquad \vec{P}\,|p,\lambda\rangle = \vec{p}\,|p,\lambda\rangle \qquad \Lambda\,|p,\lambda\rangle = \lambda\,|p,\lambda\rangle \tag{2.265}$$

and are normalized according to

$$\langle p,\lambda|p',\lambda'\rangle = (2\pi)^3\,(2E_{\vec{p}})\,\delta^3(\vec{p}-\vec{p}\,')\,\delta_{\lambda,\lambda'} \tag{2.266}$$

The collection of helicity eigenstates having 3-momentum $\vec{p}$ in the $+\hat{z}$ direction, i.e. 4-momentum $p^\mu = (E,0,0,\sqrt{E^2-M^2})$, are automatically also J_z eigenstates:

$$J_z|p',\lambda\rangle = \Lambda|p',\lambda\rangle = \lambda|p',\lambda\rangle \tag{2.267}$$

This feature allows us to derive helicity eigenstates from J_z eigenstates (and is a large part of why Section 2.6 is included in this dissertation). In doing so, we also require several other features of the helicity operator:

- **Rotations Preserve Helicity:** Because $[\Lambda,\vec{J}] = 0$, the helicity eigenvalue of a 4-momentum eigenstate is unchanged by rotations.

  Explicitly, given a generic rotation $R(\alpha)$, the 4-momentum eigenvalue will transform in the usual way, but we might expect mixing of helicity eigenvalues:

$$\mathcal{U}[R(\alpha)]|p,\lambda\rangle = e^{-i\vec{\alpha}\cdot\vec{J}}|p,\lambda\rangle = \sum_{\overline{\lambda}} c_{\overline{\lambda}}|R(\alpha)p,\overline{\lambda}\rangle \tag{2.268}$$

  where $c_{\overline{\lambda}}$ are complex coefficients. However,

$$\Lambda\mathcal{U}[R(\alpha)]|p,\lambda\rangle = \Lambda e^{-i\vec{\alpha}\cdot\vec{J}}|p,\lambda\rangle = e^{-i\vec{\alpha}\cdot\vec{J}}\Lambda|p,\lambda\rangle = \mathcal{U}[R(\alpha)]\Lambda|p,\lambda\rangle = \lambda\mathcal{U}[R(\alpha)]|p,\lambda\rangle \tag{2.269}$$

  Therefore,

$$\mathcal{U}[R(\alpha)]\,|p,\lambda\rangle \propto |R(\alpha)p,\lambda\rangle \tag{2.270}$$

  up to a phase, as desired.

- **Certain Boosts Preserve Helicity:** Because $[J^i,K^i] = 0$, the helicity eigenvalue of a 4-momentum eigenstate is unchanged by any boost along the direction of motion that preserves the 3-momentum direction.

  Consider a ket $|p,\lambda\rangle$ for which $p = (E,0,0,\sqrt{E^2-M^2})$. Under a generic boost $B_z(\beta)$ along the z-axis, the 4-momentum eigenvalue will be changed in the usual way, but the helicity eigenvalue might be changed:

$$\mathcal{U}[B_z(\beta)]|p,\lambda\rangle = e^{-i\beta K_z}|p,\lambda\rangle = \sum_{\overline{\lambda}} c_{\overline{\lambda}}|B_z(\beta)p,\overline{\lambda}\rangle \tag{2.271}$$

where $c_{\overline{\lambda}}$ are complex coefficients. Additionally suppose the boost $B_z(\beta)$ preserves the 3-momentum direction of p (so if $p' = B_z(\beta)$, then $\hat{p}' = \vec{p} = \hat{z}$), such that

$$J_z|p,\lambda\rangle = \Lambda|p,\lambda\rangle = \lambda|p,\lambda\rangle \qquad \text{and} \qquad J_z|p',\overline{\lambda}\rangle = \Lambda|p',\overline{\lambda}\rangle = \overline{\lambda}|p',\overline{\lambda}\rangle \tag{2.272}$$

Consequently, for this restricted set of kets and boosts,

$$\lambda\,\mathcal{U}[B_z(\beta)]\,|p,\lambda\rangle = \mathcal{U}[B_z(\beta)]\,\Lambda\,|p,\lambda\rangle = \mathcal{U}[B_z(\beta)]\,J_z\,|p,\lambda\rangle = J_z\,\mathcal{U}[B_z(\beta)]\,|p,\lambda\rangle \tag{2.273}$$

and

$$J_z\,\mathcal{U}[B_z(\beta)]\,|p,\lambda\rangle = \sum_{\overline{\lambda}} c_{\overline{\lambda}}\,J_z\,|B_z(\beta)p,\overline{\lambda}\rangle = \sum_{\overline{\lambda}} c_{\overline{\lambda}}\,\Lambda\,|B_z(\beta)p,\overline{\lambda}\rangle = \Lambda\,\mathcal{U}[B_z(\beta)]\,|p,\lambda\rangle \tag{2.274}$$

such that

$$\Lambda\,\mathcal{U}[B_z(\beta)]\,|p,\lambda\rangle = \lambda\,\mathcal{U}[B_z(\beta)]\,|p,\lambda\rangle \tag{2.275}$$

Therefore, so long as $B_z(\beta)$ preserves the 3-direction of p,

$$\mathcal{U}[B_z(\beta)]\,|p,\lambda\rangle \propto |B_z(\beta)p,\lambda\rangle \tag{2.276}$$

up to a phase, as desired. Note that if $|p,\lambda\rangle$ describes a massless state, then *all* boosts along the direction of motion preserve helicity.

The process of using phase conventions to eliminate proportionalities like the ones in Eqs. (2.270) and (2.276) has been handled on several occasions throughout this chapter. Specifically, Subsection 2.4.1 described the process of relating single-particle 4-momentum eigenstates on the same Lorentz-invariant hypersurface (i.e. the same mass hyperboloid or light cone). There we chose a standard 4-momentum k^μ per hypersurface with 3-momentum $\vec{k}$ pointing along the $+\hat{z}$ direction (or $\vec{k} = \vec{0}$, in the massive case). To obtain any another 4-momentum p^μ on the same Lorentz-invariant hypersurface, we boosted k^μ along the z-direction to obtain the desired 3-momentum magnitude $|\vec{p}|$ (without flipping the 3-momentum direction) and then rotated the resultant 4-momentum until its 3-momentum aimed in the desired direction as well. We now modify the massive and massless versions of this procedure to include the helicity eigenvalue.

For the massive case, the standard 4-momentum is $k^\mu = (M,0,0,0) = M\,\hat{t}^\mu$. To obtain a 4-momentum $p^\mu = E\,\hat{t}^\mu + \sqrt{E^2 - M^2}\,\hat{p}^\mu$ where $\hat{p}^\mu = (0, c_\phi s_\theta, s_\phi s_\theta, c_\theta)$, we can apply a boost and then a rotation like so:

$$p = R(\phi,\theta)B_z(\beta_{k\to p})k \qquad \text{where} \qquad \beta_{k\to p} = \operatorname{arccosh}(E_{\vec{p}}/m) \tag{2.277}$$

There are other Lorentz transformations that map k^μ to p^μ (the Lorentz group is six-dimensional whereas the mass hyperboloid is only three-dimensional), but Eq. (2.277) will be our canonical Lorentz transformation for taking k^μ to p^μ. In the quantum equivalent, we will use $|k,\lambda\rangle$ as our standard eigenket. However, we encounter an obstacle. Because $\vec{k} = \vec{0}$, the application of the helicity operator Λ to $|k,\lambda\rangle$ is not automatically well-defined:

$\Lambda|k,\lambda\rangle = (\vec{J}\cdot\vec{k})|k,\lambda\rangle/\sqrt{M^2-M^2} = (0/0)|k,\lambda\rangle$. To patch this, we modify our definition of $|k,\lambda\rangle$ and assert that k^μ should be interpreted as having an infinitesimal 3-momentum in the $+\hat{z}$ direction, such that $\Lambda|k,\lambda\rangle = J_z|k,\lambda\rangle$, thereby avoiding any reference to 3-momentum at $\vec{k}=\vec{0}$. With this solved, the quantum equivalent of the RHS of Eq. (2.277) is

$$\mathcal{U}[R(\phi,\theta)]\,\mathcal{U}[B_z(\beta_{k\to p})]\,|k,\lambda\rangle \tag{2.278}$$

We would like to use this to define single-particle states having definite 4-momentum and helicity, and thankfully we can: as previously established, the choices of $\mathcal{U}[B_z(\beta_{k\to p})]$ and $\mathcal{U}[R(\phi,\theta)]$ above preserve the helicity eigenvalue, and thus we can choose our phases such that

$$|p,\lambda\rangle \equiv \mathcal{U}[R(\phi,\theta)]\,\mathcal{U}[B_z(\beta_{k\to p})]\,|k,\lambda\rangle \tag{2.279}$$

for any massive single-particle state $|p,\lambda\rangle$. For later convenience, we define the symbol

$$|p_z,\lambda\rangle \equiv \mathcal{U}[B_z(\beta_{k\to p})]\,|k,\lambda\rangle \tag{2.280}$$

such that, for example, $|p,\lambda\rangle = \mathcal{U}[R(\phi,\theta)]|p_z,\lambda\rangle$. There remains one ambiguity in this definition, which occurs when applying Eq. (2.279) to a state with 4-momentum $-p_z \equiv (E_{\vec{p}}, -|\vec{p}|\hat{z})$. In this case, ϕ is not uniquely defined and typically does not cancel from the final result, leading to an ambiguous phase C_π that we will parameterize like so:

$$|{-p_z},\lambda\rangle \equiv C_\pi\,\mathcal{U}[R(0,\pi)]\,|p_z,\lambda\rangle \tag{2.281}$$

As per usual, setting this phase is a matter of convention. We will use the Jacob-Wick (2nd particle) convention [24, 23], which is motivated as follows: In the limit that the particle's 3-momentum vanishes, $-p_z$ and $+p_z$ both go to the rest frame 4-momentum $(m,\vec{0})$. In this same limit, the helicity operator acting on a state with 4-momentum $\pm p_z$ will go to $\pm J_z$. Therefore, up to a phase, $\lim_{|\vec{p}|\to 0}|\pm p_z,\lambda\rangle \propto |(m,\vec{0}),\pm\lambda\rangle$. Eq. (2.279) already establishes an equality in the $+p_z$ case; the Jacob-Wick convention chooses C_π so that equality will also hold in the $-p_z$ case. Because total angular momentum and helicity equal total spin and J_z in the rest frame, we can use Eq. (2.241) to find

$$\lim_{|\vec{p}|\to 0}|{-p_z},\lambda\rangle = C_\pi \lim_{|\vec{p}|\to 0}\mathcal{U}[R(0,\pi)]\,|p_z,\lambda\rangle \tag{2.282}$$

$$= C_\pi\,\mathcal{U}[R(0,\pi)]\,|(m,\vec{0}),\lambda\rangle \tag{2.283}$$

$$= C_\pi\,(-1)^{s-\lambda}\,|(m,\vec{0}),-\lambda\rangle \tag{2.284}$$

and therefore $C_\pi = (-1)^{s-\lambda}$, such that

$$|{-p_z},\lambda\rangle = (-1)^{s-\lambda}\,\mathcal{U}[R(0,\pi)]\,|p_z,\lambda\rangle \tag{2.285}$$

and this completes the construction of massive single-particle helicity eigenstates. Before moving to the massless case, we note that it is useful to define a conversion factor $\xi_\lambda(\phi)$ from the convention established in Eq. (2.279) to the Jacob-Wick convention in Eq. (2.285):

$$\xi_\lambda(\phi)\,\mathcal{U}[R(\phi,\pi)]\,|p_z,\lambda\rangle = |{-p_z},\lambda\rangle \tag{2.286}$$

or, equivalently,

$$(-1)^{s-\lambda}\,\mathcal{U}[R(\phi,-\pi)]\,\mathcal{U}[R(0,\pi)]\,|p_z,\lambda\rangle = \xi_\lambda(\phi)\,|p_z,\lambda\rangle \tag{2.287}$$

which will depend on the specific representation of the helicity eigenstates.

For the massless case, the same procedure applies in essence, but we no longer have access to a rest frame, so $\vec{k}$ cannot be made to vanish. Instead, we choose $k^\mu = E_k(\hat{t}^\mu + \hat{z}^\mu)$ for some value of energy E_k (the specific choice will not matter). Any other light-like 4-momentum $p^\mu = E(\hat{t}^\mu + \hat{p}^\mu)$ on the same lightcone can then be attained via a boost and rotation just like in Eq. (2.277), although now $\beta_{k\to p} = \ln(E_{\vec{p}}/E_{\vec{k}})$. Finally, by going over to the quantum equivalent, we can choose our phases such that Eq. (2.279) also holds for any massless eigenstate $|p,\lambda\rangle$. Recall that for non-scalar massless particles the two available helicity states are related via the reflection operator (Eq. 2.239).

Next consider Option 2, wherein our single-particle states $|E,j,m,\lambda\rangle$ satisfy

$$H\,|E,j,m,\lambda\rangle = E\,|E,j,m,\lambda\rangle \qquad \vec{J}^2\,|E,j,m,\lambda\rangle = j(j+1)\,|E,j,m,\lambda\rangle$$

$$J_z\,|E,j,m,\lambda\rangle = m\,|E,j,m,\lambda\rangle \qquad \Lambda\,|E,j,m,\lambda\rangle = \lambda\,|E,j,m,\lambda\rangle \tag{2.288}$$

Because $P^2 = H^2 - \vec{P}^{\,2}$, each state $|E,j,m,\lambda\rangle$ is also an eigenstate of $\vec{P}^{\,2}$ with eigenvalue $E^2 - M^2$. As a result, these states are sometimes labeled by $|\vec{p}| = \sqrt{E^2-M^2}$ in place of E in the literature.

Using properties of the above definitions and properties of the Wigner D matrix, we may write:

$$|p,\lambda\rangle = \sum_{j,m}\sqrt{\frac{2j+1}{4\pi}}\mathcal{D}^j_{m,\lambda}(\phi,\theta)|E,j,m,\lambda\rangle \tag{2.289}$$

This defines the single-particle state $|p,\lambda\rangle$ in terms of the angular momentum eigenstates $|E,j,m,\lambda\rangle$ [24, 23]. Inverting this yields,

$$|E,j,m,\lambda\rangle = \sqrt{\frac{2j+1}{4\pi}}\int d\Omega\;D^{j*}_{m\lambda}(\phi,\theta)|p,\lambda\rangle \tag{2.290}$$

As in Subsections 2.4.2 and 2.4.3, we can combine single-particle states to form multi-particle states. If we follow that procedure, we would define a (distinguishable) two-particle state as

$$|p_1,\lambda_1\rangle \otimes |p_2,\lambda_2\rangle \tag{2.291}$$

where each single-particle state is defined according to Eqs. (2.279) and (when $\vec{p} = -|\vec{p}|\hat{z}$) (2.285). However when considering two-particle states in the center-of-momentum frame, this is not the convention typically adopted.

Instead, it conventional to define the two-particle COM states as

$$|\vec{p},\lambda_1,\lambda_2\rangle \equiv \Big(\mathcal{U}[R(\phi,\theta)]|(E_1,+|\vec{p}\,|\hat{z}),\lambda_1\rangle\Big) \otimes \Big(\mathcal{U}[R(\phi,\theta)]|(E_2,-|\vec{p}\,|\hat{z}),\lambda_2\rangle\Big) \tag{2.292}$$

This is why the phase convention for $|-p_z, \lambda\rangle$ chosen in Eq. (2.285) for single-particle states is typically called the Jacob-Wick 2nd particle convention. We also define the two-particle total and relative helicity operators as $\Lambda_{\text{total}} = \Lambda_1 + \Lambda_2$ and $\Lambda = \Lambda_1 - \Lambda_2$ respectively, where

$$\Lambda_1 \pm \Lambda_2 = \frac{\vec{J}_1 \cdot \vec{P}_1}{\sqrt{E_1^2 - m_1^2}} \pm \frac{\vec{J}_2 \cdot \vec{P}_2}{\sqrt{E_2^2 - m_2^2}} \underset{\text{frame}}{\overset{\text{COM}}{=}} (\vec{J}_1 \mp \vec{J}_2) \cdot \hat{p} \tag{2.293}$$

and the last equality in each line assumes it acts on a state with definite 3-momentum $\vec{p}$. Note that the relative helicity is related to the two-particle angular momentum operator $\vec{J} = \vec{J}_1 + \vec{J}_2$.

The single-particle argument that allowed $|p, \lambda\rangle$ to be rewritten as a superposition of $|E, j, m, \lambda\rangle$ carries through essentially unchanged for $|\vec{p}, \lambda_1, \lambda_2\rangle$ in terms of the relative helicity $\lambda = \lambda_1 - \lambda_2$, such that we may write the state $|\vec{p}, \lambda_1, \lambda_2\rangle$ in terms of two-particle angular momentum eigenstates as

$$|\vec{p}, \lambda_1, \lambda_2\rangle = \sum_{J,M} \sqrt{\frac{2j+1}{4\pi}}\, \mathcal{D}^{J}_{M,\lambda_1-\lambda_2}(\phi, \theta)\, |\sqrt{s}, J, M, \lambda_1, \lambda_2\rangle \tag{2.294}$$

Because they occur regularly in 2-to-2 scattering calculations, the relative helicities of the initial and final particle pairs are given special symbols: $\lambda_i \equiv \lambda_1 - \lambda_2$ and $\lambda_f \equiv \lambda_3 - \lambda_4$.

2.7.2 Partial Wave Amplitudes

Because the S-matrix commutes with the total angular momentum operator $\vec{J}^2$, it can be put into block-diagonal form wherein each block has a definite total angular momentum. This implies a similar decomposition of the T-matrix,

$$\langle\sqrt{s}, J, M, \lambda_3, \lambda_4|\hat{T}|\sqrt{s}', J', M', \lambda_1, \lambda_2\rangle \equiv \delta_{J',J}\, \delta_{M',M}\, \langle\lambda_3, \lambda_4|\, \hat{T}^J(s)\, |\lambda_1, \lambda_2\rangle \tag{2.295}$$

such that, via Eq. (2.294) and because $\mathcal{D}^{j}_{m_1,m_2}(\phi, 0) = \delta_{m_1,m_2}$,

$$\mathcal{M}_{i\to f} = \langle\vec{p}, \lambda_3, \lambda_4|\hat{T}|\lambda_1, \lambda_2\rangle \tag{2.296}$$

$$= \sum_{J,M,J',M'} \sqrt{\frac{2J+1}{4\pi}}\sqrt{\frac{2J'+1}{4\pi}}\, \mathcal{D}^{J*}_{M,\lambda_f}(\phi, \theta)\, \mathcal{D}^{J'}_{M',\lambda_i}(0, 0) \times \langle\sqrt{s}, J, M, \lambda_3, \lambda_4|\, \hat{T}(s)\, |\sqrt{s}', J', M', \lambda_1, \lambda_2\rangle \tag{2.297}$$

$$= \sum_{J} \left(\frac{2J+1}{4\pi}\right) \mathcal{D}^{J*}_{\lambda_i,\lambda_f}(\phi, \theta)\, \langle\lambda_3, \lambda_4|\, \hat{T}^J(s)\, |\lambda_1, \lambda_2\rangle \tag{2.298}$$

where $\lambda_i = \lambda_1 - \lambda_2$ and $\lambda_f = \lambda_3 - \lambda_4$. The partial wave amplitude is defined as

$$a^J(s) \equiv \frac{1}{32\pi^2}\langle\lambda_3, \lambda_4|\, \hat{T}^J(s)\, |\lambda_1, \lambda_2\rangle \tag{2.299}$$

In terms of the partial wave amplitudes, Eq. (2.298) becomes

$$\mathcal{M}(s,\theta,\phi) = \sum_J 8\pi(2J+1)\, a^J(s)\, D^{J*}_{\lambda_i\lambda_f}(\theta,\phi) \tag{2.300}$$

In the next subsection, we utilize this decomposition of the 2-to-2 scattering matrix element into partial wave amplitudes in order to derive the elastic and inelastic partial wave unitarity constraints from the optical theorem.

2.7.3 Elastic, Inelastic Unitarity Constraints

Recall Eq. (2.212), wherein we reduced the optical theorem to

$$2\Im[\mathcal{M}_{i\to i}] = \sum_{f_2} \frac{\mathbb{P}(3,4)}{16\pi^2 E_i} \int d\Omega \quad |\mathcal{M}_{i\to f_2}|^2 + C_{f\neq f_2} \tag{2.301}$$

Using Eq. (2.300), decompose the matrix element on the RHS of Eq. (2.301), such that

$$\int d\Omega \quad |\mathcal{M}_{i\to f_2}|^2 = \mathcal{M}^*_{i\to f_2}\mathcal{M}_{i\to f_2} \tag{2.302}$$

$$= \int d\Omega \quad \left[\sum_{J'} 8\pi(2J'+1)\, a^{J'*}_{i\to f_2}(s)\, D^{J'}_{\lambda_i\lambda_f}(\theta,\phi)\right] \cdot \left[\sum_J 8\pi(2J+1)\, a^J_{i\to f_2}(s)\, D^{J*}_{\lambda_i\lambda_f}(\theta,\phi)\right] \tag{2.303}$$

$$= \sum_J 256\pi^3(2J+1)|a^J_{i\to f_2}(s)|^2 \tag{2.304}$$

and overall the RHS of Eq. (2.301) becomes

$$\sum_{f_2}\sum_J 16\pi(2J+1)\frac{\mathbb{P}(3,4)}{E_i}|a^J_{i\to f_2}(s)|^2 + C_{i\neq f} \tag{2.305}$$

In the same frame, matrix element on the LHS of Eq. (2.301) equals

$$\mathcal{M}_{i\to i} = \sum_J 8\pi(2J+1)\, a^J_{i\to i}(s)\, D^{J*}_{\lambda_i\lambda_i}(0,0) = \sum_J 8\pi(2J+1)\, a^J_{i\to i}(s) \tag{2.306}$$

such that the LHS equals, overall,

$$2\Im[\mathcal{M}_{i\to i}] = \sum_J 16\pi(2J+1)\Im[a^J_{i\to i}(s)] \tag{2.307}$$

and all together Eq. (2.301) implies

$$\sum_J 16\pi(2J+1)\Im[a^J_{i\to i}(s)] = \sum_{f_2}\sum_J 16\pi(2J+1)\frac{\mathbb{P}(3,4)}{E_i}|a^J_{i\to f_2}(s)|^2 + C_{i\neq f} \tag{2.308}$$

or, focusing on the 2-to-2 scattering and dropping the nonnegative constant $C_{i\neq f}$,

$$\sum_J (2J+1)\Im[a^J_{i\to i}(s)] > \sum_{f_2}\sum_J (2J+1)\frac{\mathbb{P}(3,4)}{E_i}|a^J_{i\to f_2}(s)|^2 \tag{2.309}$$

We can isolate individual angular momentum components by employing superpositions of helicity eigenstates that reconstruct the angular momentum eigenstates, such that

$$\Im[a^J_{i\to i}(s)] > \sum_{f_2}\frac{\mathbb{P}(3,4)}{E_i}|a^J_{i\to f_2}(s)|^2 \tag{2.310}$$

The RHS of this inequality can be further reduced by dividing the expression into elastic ($i = f_2$, aside from the values of (θ,ϕ) describing the pair) and inelastic ($i \neq f_2$) pieces,

$$\Im[a^J_{i\to i}(s)] > \frac{\mathbb{P}(1,2)}{E_i}|a^J_{i\to i}(s)|^2 + \sum_{f_2\neq i}\frac{\mathbb{P}(3,4)}{E_i}|a^J_{i\to f_2}(s)|^2 \tag{2.311}$$

By definition, $|a^J_{i\to i}(s)|^2 = \Re[a^J_{i\to i}(s)]^2 + \Im[a^J_{i\to i}(s)]^2$, so the previous inequality can also be expressed as, after multiplying both sides by $\mathbb{P}(1,2)/E_i$ and rearranging,

$$1 - \sum_{f_2\neq i}\beta_{12}\,\beta_{34}\,|a^J_{i\to f_2}(s)|^2 > \left[\beta_{12}\,\Re[a^J_{i\to i}(s)]\right]^2 + \left[\beta_{12}\,\Im[a^J_{i\to i}(s)] - 1\right]^2 \tag{2.312}$$

where

$$\beta_{jk} \equiv 2\frac{\mathbb{P}(j,k)}{E_i} = \frac{1}{s}\sqrt{\left[s-(m_j-m_k)^2\right]\left[s-(m_j+m_k)^2\right]} \tag{2.313}$$

because $E_i = E_1 + E_2 = \sqrt{s}$. Thus, the values of $\beta_{12}a^J_{i\to i}(s)$ are bounded by a circle in the complex plane centered at i and with radius at most equal to 1. Therefore, the real and imaginary parts of the elastic amplitudes must satisfy

$$\left|\beta_{12}\,\Re[a^J_{i\to i}]\right| \leq 1 \qquad\qquad 0 \leq \beta_{12}\,\Im[a^J_{i\to i}] \leq 2 \tag{2.314}$$

Meanwhile, the RHS of Eq. (2.312) must be nonnegative, so the net sum of squares of inelastic amplitudes are bounded from above

$$\sum_{f_2\neq i}\beta_{12}\beta_{34}|a^J_{i\to f_2}(s)|^2 < 1 \tag{2.315}$$

These are the inequalities we sought to derive [25]. For most of the processes in which we're interested, $\mathcal{M}$ grows like $\mathcal{O}(s^k)$ at large s for $k \geq 1$, such that (via Eq. (2.300)) $a^J(s) \sim \mathcal{O}(s^k)$ as well. If these inequalities are satisfied for such a partial wave amplitude at some value of s, then there necessarily exists an energy scale Λ_{strong} for which all $s \geq \Lambda_{\text{strong}}$

contradict these inequalities, and thus contradict the optical theorem, and thus contradict unitarity of the S-matrix.

An additional factor of 1/2 should be included in β_{jk} if the particles associated with it are identical, per the discussion at the end of Subsection 2.4.3. For an elastic process ($i = f$) wherein the initial (and thus final) particles are identical,

$$\beta_{11} = \frac{1}{2}\sqrt{1 - \frac{4m_1^2}{s}} \tag{2.316}$$

such that the relevant partial wave unitarity constraints are

$$\sqrt{1 - \frac{s_{\text{min}}}{s}}\ \left|\Re[a^J_{i\to i}]\right| \leq 2 \qquad\qquad 0 \leq \sqrt{1 - \frac{s_{\text{min}}}{s}}\ \left|\Re[a^J_{i\to i}]\right| \leq 4 \tag{2.317}$$

where $s_{\text{min}} = 4m_1^2$.

2.8 Polarization Tensors and Lagrangians

2.8.1 Derivation of the Spin-1 and Spin-2 Polarizations

We mentioned in Subsection 2.2.3 that the 4-vector representation of the Lorentz group embeds spin-0 and spin-1 representations. This subsection now derives the spin-1 representation, and then uses Clebsch-Gordan coefficients (refer to Subsection 2.6.2) to combine two copies of the spin-1 representation to form a spin-2 representation. The end product of this procedure are the spin-1 and spin-2 polarization structures, which accompany external states when calculating matrix elements.

To obtain these structures, we must generalize the 4-vector representation described in Section 2.2: we promote the 4-vector generators to quantum generators $J^i = i(J^i)_{\text{4-vector}}$ and $K^i = i(K^i)_{\text{4-vector}}$ so that a generic rotation and boost equal $R(\vec{\alpha}) = \text{Exp}[-i\vec{\alpha}\cdot\vec{J}]$ with generators

$$J^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \end{pmatrix} \qquad J^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \qquad J^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & +i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.318}$$

and

$$K^1 = \begin{pmatrix} 0 & +i & 0 & 0 \\ +i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K^2 = \begin{pmatrix} 0 & 0 & +i & 0 \\ 0 & 0 & 0 & 0 \\ +i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K^3 = \begin{pmatrix} 0 & 0 & 0 & +i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ +i & 0 & 0 & 0 \end{pmatrix} \tag{2.319}$$

with commutators $[J^i, J^j] = +i\epsilon_{ijk}J^k$, $[J^i, K^j] = +i\epsilon_{ijk}K^k$, and $[K^i, K^j] = -i\epsilon_{ijk}J^k$. We do not include a $\mathcal{U}$ label on the rotation and boost transformations in this representation because componentwise they are exactly the same as before the quantum promotion, and will act on quantum states having 4-vector indices. As such, our kets may be written as ϵ^μ and transform under a Lorentz transformation Λ according to $\Lambda^\mu{}_\nu \epsilon^\nu$. That being said, by going over to the quantum equivalent we now work in the space of *complex* 4-vectors.

Suppose the complex 4-vectors ϵ^μ encode single-particle states with definite 4-momentum p and helicity λ, i.e. there exists a 4-vector basis $\epsilon^\mu_{s,\lambda}(p)$ (analogous to the kets $|p\lambda\rangle$ defined in Section 2.4 with the internal spin s explicitly indicated). We can construct these states explicitly by using the techniques explained in Subsection 2.7.1, wherein a standard 4-momentum k^μ per Lorentz-invariant hypersurface is used to define any other state having 4-momentum p^μ on that same hypersurface.

For a single-particle state with nonzero mass m, consider $\epsilon^\mu_{s,\lambda}(p)$ in the rest frame (corresponding to the standard 4-momentum $k^\mu = (m, \vec{0})$). In this frame, the helicity operator $\Lambda = (\vec{J} \cdot \vec{P})/\sqrt{E^2 - m^2}$ reduces to $J_z = J^3$, so we can find the helicity eigenstates $\epsilon^\mu_\lambda(k)$ by finding J_z eigenstates. Note that the total angular momentum operator in this representation equals

$$\vec{J}^2 \equiv \vec{J} \cdot \vec{J} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & +2 & 0 & 0 \\ 0 & 0 & +2 & 0 \\ 0 & 0 & 0 & +2 \end{pmatrix} \tag{2.320}$$

Because $\vec{J}^2$ has eigenvalues of the form $j(j+1)$ in general, we recognize a $j = 0$ (with $m = 0$) and a $j = 1$ representation encoded in $\vec{J}^2$. Thankfully, $\vec{J}^2$ is block-diagonal as-is, so we can directly construct projection operators $\mathfrak{P}_0(k)$ and $\mathfrak{P}_1(k)$ that (when acted on a generic complex 4-vector in the rest frame) will isolate the $j = 0$ and $j = 1$ representations therein:

$$[\mathfrak{P}_0(k)^\mu{}_\nu] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad [\mathfrak{P}_1(k)^\mu{}_\nu] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.321}$$

or, in terms of the 4-momentum k^μ and the Minkowski metric $\eta_{\mu\nu}$,

$$[\mathfrak{P}_0(k)^\mu{}_\nu] = \frac{k^\mu k_\nu}{m^2} \qquad [\mathfrak{P}_1(k)^\mu{}_\nu] = \eta^\mu{}_\nu - \frac{k^\mu k_\nu}{m^2} \tag{2.322}$$

We will use these to obtain the rest frame helicity eigenstates.

For example, to find the $j = 0$ helicity eigenstate in the rest frame, we act the spin-0 projection operator $\mathfrak{P}_0(k)$ on a generic complex 4-vector $\epsilon^\mu(k)$ in the rest frame and then solve for eigenstates of $\Lambda = J_z$, e.g. we solve $J_z\,[\mathfrak{P}_0(k)\,\epsilon(k)] = \lambda\,[\mathfrak{P}_0(k)\,\epsilon(k)]$ for the available helicities. When $j = 0$, the only helicity available is $\lambda = 0$. This yields $\epsilon^\mu_{0,0} \propto k^\mu/m$ up to a phase. However, this representation has little use in actual quantum field theory calculations because there exist a more succinct Lorentz covariant spin-0 representation: the Lorentz scalar $\epsilon_0(k) = 1$. Thus, we consider the $j = 0$ part of this representation no further, and simply write $\epsilon^\mu_\lambda(p)$ instead of $\epsilon^\mu_{1,\lambda}(p)$ as is conventional.

The process for finding the $j = 1$ helicity eigenstates in the rest frame is essentially identical to the $j = 0$ case: we aim to solve $J_z\,[\mathfrak{P}_1(k)\,\epsilon(k)] = \lambda\,[\mathfrak{P}_1(k)\,\epsilon(k)]$ for helicities $\lambda \in \{-1, 0, +1\}$. For example, when $\lambda = 0$,

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ \epsilon^1(k) \\ \epsilon^2(k) \\ \epsilon^3(k) \end{pmatrix} = \lambda\,[(\mathfrak{P}_1(k)\,\epsilon(k))^\mu] = [(J_z\,\mathfrak{P}_1(k)\,\epsilon(k))^\mu] = \begin{pmatrix} 0 \\ -i\epsilon^2(k) \\ i\epsilon^1(k) \\ 0 \end{pmatrix} \tag{2.323}$$

such that $\epsilon^1(k) = \epsilon^2(k) = 0$, and

$$[\epsilon_0^\mu(k)] \quad = \quad [(\mathfrak{P}_1(k)\,\epsilon(k))^\mu] \text{ with helicity } \lambda = 0 \quad \propto \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ \epsilon^3(k) \end{pmatrix} \tag{2.324}$$

up to a to-be-determined normalization and phase. Note:

$$\epsilon_0(k) \cdot \epsilon_0(k) = -\epsilon^3(k)^2 \tag{2.325}$$

It is conventional to set this magnitude to -1 by choosing the aforementioned phase such that $\epsilon_0^\mu(k) = (0, 0, 0, 1)$.

The $\lambda = \pm 1$ solutions can subsequently be obtained via the ladder operators

$$J_\pm = J_x \pm iJ_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp 1 \\ 0 & 0 & 0 & -i \\ 0 & \pm 1 & +i & 0 \end{pmatrix} \tag{2.326}$$

and Eq. (2.234), such that (noting $\sqrt{(j \mp m)(j \pm m + 1)} \to \sqrt{j(j+1)} = \sqrt{2}$ in this case)

$$[\epsilon_{\pm 1}(k)^\mu] = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp 1 \\ 0 & 0 & 0 & -i \\ 0 & \pm 1 & +i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \mp 1 \\ -i \\ 0 \end{pmatrix} \tag{2.327}$$

The polarization vectors $\{\epsilon_{-1}^\mu(k), \epsilon_0^\mu(k), \epsilon_{+1}^\mu(k)\}$ form the desired $j = 1$ representation in the rest frame. Explicit calculation reveals they are orthonormal and transverse,

$$\epsilon_\lambda(k)^* \cdot \epsilon_{\lambda'}(k) = -\delta_{\lambda,\lambda'} \qquad\qquad k \cdot \epsilon_\lambda(k) = 0 \tag{2.328}$$

and as a basis for the $j = 1$ representation they naturally resolve the projection operator $\mathfrak{P}_1(k)$, which is the identity on the $j = 1$ subspace:

$$[\mathfrak{P}_1(k)^{\mu\nu}] = -\sum_{\lambda=-1}^{+1} \epsilon_\lambda^\mu(k)^* \epsilon_\lambda^\nu(k) \tag{2.329}$$

This completes the derivation of the $j = 1$ representation in the rest frame.

To obtain this representation in all other frames, we apply the standard Lorentz transformation $\Lambda_{k\to p} = R(\phi,\theta)\, B_z(\beta_{k\to p})$ defined in Eq. (2.277) of Section 2.7 to each polarization vector $\epsilon^\mu_\lambda(k)$, where we define

$$\epsilon^\mu_\lambda(k) \equiv (\Lambda_{k\to p})^\mu{}_\nu\, \epsilon^\mu_\lambda(p) \tag{2.330}$$

Note that the internal spin of a particle corresponds to a Casimir operator of the Lorentz group, so it is invariant under $\Lambda_{k\to p}$, and the $j = 0$ and $j = 1$ representations do not mix. In the 4-vector representation,

$$[(\Lambda_{k\to p})^\mu{}_\nu] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\phi^2 c_\theta + s_\phi^2 & c_\phi s_\phi (c_\theta - 1) & c_\phi s_\theta \\ 0 & c_\phi s_\phi (c_\theta - 1) & c_\phi^2 + c_\theta s_\phi^2 & s_\phi s_\theta \\ 0 & -c_\phi s_\theta & -s_\phi s_\theta & c_\theta \end{pmatrix} \frac{1}{m} \begin{pmatrix} E & 0 & 0 & \vec{p} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \vec{p} & 0 & 0 & E \end{pmatrix} \tag{2.331}$$

such that

$$[\epsilon^\mu_{\pm 1}(p)] = \pm \frac{e^{\pm i\phi}}{\sqrt{2}} \begin{pmatrix} 0 \\ -c_\theta c_\phi \pm i s_\phi \\ -c_\theta s_\phi \mp i c_\phi \\ s_\theta \end{pmatrix} \qquad [\epsilon^\mu_0(p)] = \frac{1}{m} \begin{pmatrix} |\vec{p}| \\ E_{\vec{p}}\, c_\phi s_\theta \\ E_{\vec{p}}\, s_\phi s_\theta \\ E_{\vec{p}}\, c_\theta \end{pmatrix} = \frac{1}{m} \begin{pmatrix} |\vec{p}| \\ E_{\vec{p}}\hat{p} \end{pmatrix} \tag{2.332}$$

where $\hat{p} = \hat{z}$ when $\vec{p} = \vec{0}$, per the helicity eigenstate convention established in Subsection 2.7.1. Note that the helicity-zero polarization tensor grows like $\mathcal{O}(E)$, whereas the others do not depend on energy at all. Because of Lorentz covariance, the spin-1 polarization vectors $\epsilon^\mu_\lambda(p)$ retain their rest frame properties (orthogonal, transverse),

$$\epsilon_\lambda(p)^* \cdot \epsilon_{\lambda'}(p) = -\delta_{\lambda,\lambda'} \qquad p \cdot \epsilon_\lambda(p) = 0 \tag{2.333}$$

and the $j = 1$ projection operator becomes

$$[\mathfrak{P}_1(k)^{\mu\nu}] = -\sum_{\lambda=-1}^{+1} \epsilon^\mu_\lambda(k)^* \epsilon^\nu_\lambda(k) = \eta^{\mu\nu} - \frac{p^\mu p^\nu}{m^2} \tag{2.334}$$

In this representation, the reflection operator equals $Y = \mathrm{Diag}(1, 1, -1, 1)$, such that

$$\begin{cases} Y^\mu{}_\nu\, \epsilon^\nu_0(p_z) = \epsilon^\mu_0(p_z) \\ Y^\mu{}_\nu\, \epsilon^\nu_{\pm 1}(p_z) = -\epsilon^\mu_{\mp 1}(p_z) \end{cases} \qquad \Longrightarrow \qquad Y^\mu{}_\nu\, \epsilon^\nu_\lambda(p_z) = -(-1)^{1-\lambda} \epsilon^\mu_{-\lambda}(p_z) \tag{2.335}$$

This completes the derivation of the massive spin-1 polarization vectors.

The Jacob-Wick 2nd particle conversion factor $\xi_\lambda(\phi)$ from Eq. (2.286) can be calculated directly. First note that,

$$[R(\phi, -\pi)^\mu{}_\nu R(0, \pi)^\nu{}_\rho] = \begin{pmatrix} 1 & 0 & 0 & \vec{p} \\ 0 & +c_{2\phi} & -s_{2\phi} & 0 \\ 0 & +s_{2\phi} & +c_{2\phi} & 0 \\ \vec{p} & 0 & 0 & 1 \end{pmatrix} \tag{2.336}$$

such that

$$\left[(-1)^{1-\lambda}\, R(\phi,-\pi)^{\mu}{}_{\nu}\, R(0,\pi)^{\nu}{}_{\rho}\right] \epsilon^{\rho}_{\lambda}(p_z) = (-1)^{1-\lambda}\, e^{-2\lambda i\phi}\, \epsilon^{\rho}_{\lambda}(p_z) \tag{2.337}$$

and finally $\xi_\lambda(\phi) = (-1)^{1-\lambda} e^{-2\lambda i\phi}$.

The derivation of the massless spin-1 polarization vectors follows the same trajectory, but now there is no rest frame and their helicities are restricted to $\lambda = \pm 1$. However, we already have helicity eigenstates corresponding to $\lambda = \pm 1$ which work in any frame, and sure enough the polarization vectors $\epsilon^{\mu}_{\pm 1}(p)$ are admissible helicity eigenstates for massless spin-1 particles.

Onward to the spin-2 helicity eigenstates. As described in Subsection 2.6.2, any two angular momentum representations can be combined to form a new angular momentum representation via the Clebsch-Gordan coefficients. Thus, we can combine two copies of our (massive or massless) spin-1 polarization vectors $\epsilon^{\mu}_{\lambda}(p)$ and thereby obtain a Lorentz-covariant representation of spin-2 particles via polarization tensors $\epsilon^{\mu\nu}_{\lambda}(p)$. Explicitly, the spin-2 polarization tensors equal, using Eq. (2.255),

$$\epsilon^{\mu\nu}_{\pm 2}(p) = \epsilon^{\mu}_{\pm 1}(p)\, \epsilon^{\nu}_{\pm 1}(p)\ , \tag{2.338}$$

$$\epsilon^{\mu\nu}_{\pm 1}(p) = \frac{1}{\sqrt{2}}\left[\epsilon^{\mu}_{\pm 1}(p)\, \epsilon^{\nu}_{0}(p) + \epsilon^{\mu}_{0}(p)\, \epsilon^{\nu}_{\pm 1}(p)\right] \tag{2.339}$$

$$\epsilon^{\mu\nu}_{0} = \frac{1}{\sqrt{6}}\left[\epsilon^{\mu}_{+1}(p)\, \epsilon^{\nu}_{-1}(p) + \epsilon^{\mu}_{-1}(p)\, \epsilon^{\nu}_{+1}(p) + 2\epsilon^{\mu}_{0}(p)\, \epsilon^{\nu}_{0}(p)\right]\ , \tag{2.340}$$

where the massive case has access to all five helicity states ($\lambda = \pm 2, \pm 1, 0$) and the massless case only has access to two ($\lambda = \pm 2$). Via the properties of the polarization vectors that compose them, each polarization tensor is traceless, symmetric, and transverse:

$$\eta_{\mu\nu}\epsilon^{\mu\nu}_{\lambda}(p) = 0 \qquad \epsilon^{\mu\nu}_{\lambda}(p) = \epsilon^{\nu\mu}_{\lambda}(p) \qquad p_\mu \epsilon^{\mu\nu}_{\lambda}(p) = 0 \tag{2.341}$$

By applying the appropriate generalization of the helicity reflection operator $Y^{\mu\nu}{}_{\rho\sigma} = Y^{\mu}{}_{\rho} Y^{\nu}{}_{\sigma}$, we find

$$Y^{\mu\nu}{}_{\rho\sigma}\, \epsilon^{\rho\sigma}_{\lambda}(p_z) = (-1)^{2-\lambda} \epsilon^{\mu\nu}_{-\lambda}(p_z) \tag{2.342}$$

Finally, the spin-2 Jacob-Wick 2nd particle conversion factor can be determined by applying the spin-1 conversion factor to each spin-1 polarization vector in the definitions of the spin-2 polarization tensor, thereby yielding $\xi_\lambda(\phi) = (-1)^{2-\lambda} e^{-2\lambda i\phi}$.

2.8.2 Quadratic Lagrangians and Propagators

This chapter has largely focused on the construction of external particle states as 4-momentum and helicity eigenstates. In order to calculate matrix elements describing scattering processes between these external states, we must encode those external states into quantum fields and use those quantum fields to construct Lagrangians. The quadratic terms of a Lagrangian determine the masses and spins of the particles encoded within the fields, whereas higher-order terms of a Lagrangian determine interactions between various particles.

Perhaps the simplest field and Lagrangian corresponds to a spin-0 massless particle. A field $\hat{r}(x)$ will encode (real) massless spin-0 particles if our overall Lagrangian possesses the quadratic terms

$$\mathcal{L}^{(s=0)}_{\text{massless}} \equiv \frac{1}{2}(\partial_\mu \hat{r})^2 \tag{2.343}$$

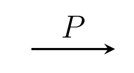
To derive the propagator associated with this Lagrangian, we

- Fourier transform to 4-momentum space, effectively replacing ∂_μ with $-iP_\mu$, where P_μ is the 4-momentum carried through the propagator,
- Take the functional derivative with respect to the field twice, and
- Invert the resulting expression, and multiply by $-i$

Applying this procedure to Eq. (2.345) yields

$$\mathcal{L}^{(s=0)}_{\text{massless}} \quad \rightarrow \quad -\frac{1}{2}P^2 \hat{r}^{\,2} \quad \rightarrow \quad -P^2 \quad \rightarrow \quad \frac{i}{P^2} \tag{2.344}$$

and, thus, we find the (momentum space) massless spin-0 propagator equals

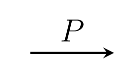


$$= \frac{i}{P^2}$$

If we instead desire a (real) *massive* spin-0 field $\hat{r}(x)$, we can add a mass term $-(1/2)M^2\hat{r}^{\,2}$ to the massless spin-0 Lagrangian:

$$\mathcal{L}^{(s=0)}_{\text{massless}} \equiv \frac{1}{2}(\partial_\mu \hat{r})^2 - \frac{1}{2}M^2\hat{r}^2 \tag{2.345}$$

in which case the same procedure instead yields



$$= \frac{i}{P^2 - M^2}$$

As derived by Fierz and Pauli [20, 26], the canonical massless spin-2 quadratic Lagrangian is

$$\mathcal{L}^{(s=2)}_{\text{massless}} \equiv (\partial\hat{h})_\mu(\partial^\mu \hat{h}) - (\partial\hat{h})^2_\mu + \frac{1}{2}(\partial_\mu \hat{h}_{\nu\rho})^2 - \frac{1}{2}(\partial_\mu \hat{h})^2 \tag{2.346}$$

Unfortunately, we cannot directly apply the previous procedure to obtain the massless spin-2 propagator because in the course of embedding a massless spin-2 particle (with two degrees of freedom) into a rank-2 symmetric traceless Lorentz tensor $\hat{h}_{\mu\nu}$ (with five degrees of freedom) so that we could rely on Lorentz covariance, we introduced gauge redundancies. This gauge freedom makes the differential operator defined in Eq. (2.346) non-invertible. In particular, the massless spin-2 Lagrangian is unchanged by the following gauge transformation:

$$\hat{h}_{\mu\nu} \quad \longrightarrow \quad \hat{h}_{\mu\nu} + (\partial_\mu \epsilon_\nu) + (\partial_\nu \epsilon_\mu) \tag{2.347}$$

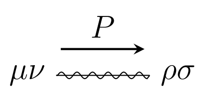
for a generic 4-vector field $\epsilon_\mu(x)$. In fact, Eq. (2.346) is the only combination of quadratic-level kinetic terms for $\hat{h}_{\mu\nu}$ that is invariant under this gauge transformation, such that we could have started by demanding invariance under transformations of the form Eq. (2.347) and thereby derived $\mathcal{L}^{(s=2)}_{\text{massless}}$. In order to invert Eq. (2.346) and obtain a massless spin-2 propagator, we must somehow break this gauge invariance. This can be done in a multitude of ways, whether it be by employing a specific gauge condition or adding a gauge-fixing term to the Lagrangian. A popular gauge choice is the harmonic gauge, which is defined by setting

$$\partial^\mu \hat{h}^{(0)}_{\mu\nu} = \tfrac{1}{2}\partial_\nu [\![\hat{h}^{(0)}]\!] \tag{2.348}$$

This isolates a specific gauge orbit, thereby breaking the gauge invariance of the quadratic Lagrangian Eq. (2.346) and allowing it to be inverted into a propagator. However, this dissertation does not use harmonic gauge (or any other gauge condition), instead opting to add a gauge-fixing term $\mathcal{L}_{\text{gf}}$ to the massless spin-2 Lagrangian. Specifically, we employ the de Donder gauge, which has a gauge-fixing term

$$\mathcal{L}_{\text{gf}} \equiv -\left(\partial^\mu \hat{h}_{\mu\nu} - \frac{1}{2}\partial_\nu \hat{h}\right)^2 \tag{2.349}$$

Rather than isolate any single gauge orbit, de Donder gauge averages over a continuum of gauge orbits. This averaging is weighted in favor of the harmonic gauge condition, the bias of which successfully breaks the troublesome gauge invariance of Eq. (2.346). The resulting de Donder gauge massless spin-2 propagator equals



$$= \frac{iB_0^{\mu\nu,\rho\sigma}}{P^2}$$

where

$$B_0^{\mu\nu,\rho\sigma} \equiv \frac{1}{2}\left[\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}\right] \tag{2.350}$$

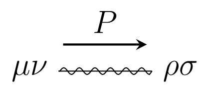
In the same way that we went from massless to massive spin-0 Lagrangian, the massive spin-2 Lagrangian is obtained from the massless spin-2 Lagrangian (without the gauge-fixing term) by adding a mass term. As it turns out, there is only one non-kinetic quadratic combination of the field $\hat{h}_{\mu\nu}$ which yields a propagator pole at $P^2 = M^2$ and does not introduce ghosts [20]. This combination defines the Fierz-Pauli mass terms,

$$\mathcal{L}_{\text{FP}}(m, \hat{h}) \equiv m^2\left[\frac{1}{2}\hat{h}^2 - \frac{1}{2}[\![\hat{h}\hat{h}]\!]\right] \tag{2.351}$$

which when added to the massless spin-2 Lagrangian yields the canonical massive spin-2 quadratic Lagrangian:

$$\mathcal{L}^{(s=2)}_{\text{massive}} \equiv \mathcal{L}^{(s=2)}_{\text{massless}} + m^2\left[\frac{1}{2}\hat{h}^2 - \frac{1}{2}[\![\hat{h}\hat{h}]\!]\right] \tag{2.352}$$

Because the Fierz-Pauli mass term breaks the gauge invariance of the massless Lagrangian, all five degrees of freedom in the symmetric traceless field $\hat{h}_{\mu\nu}$ can propagate, which is in agreement with the five helicity states we expect from a massive spin-2 particle. This also allows us to invert $\mathcal{L}^{(s=2)}_{\text{massive}}$ and obtain the massive spin-2 propagator:



$$= \frac{iB^{\mu\nu,\rho\sigma}}{P^2 - M^2}$$

where

$$B^{\mu\nu,\rho\sigma} = \frac{1}{2}\left[\overline{B}^{\mu\rho}\overline{B}^{\nu\sigma} + \overline{B}^{\mu\sigma}\overline{B}^{\nu\rho} - \frac{2}{3}\overline{B}^{\mu\nu}\overline{B}^{\rho\sigma}\right] \tag{2.353}$$

This is the last piece of four-dimensional quantum field theory information that we require for calculating the desired scattering amplitudes. In the next chapter, we introduce the necessary information about five-dimensional field theories, including the machinery of general relativity machinery and the definition of the Randall Sundrum 1 model.

Chapter 3

The 5D RS1 Model

3.1 Chapter Summary

The previous chapter introduced the important definitions and conventions from 4D quantum field theory, including discussions of 2-to-2 scattering, helicity eigenstates, and partial wave unitarity constraints. It also defined the twice-squared bracket notation which is used often throughout the remainder of this dissertation: given a collection of spin-2 fields $\{\hat{h}^{(1)}, \hat{h}^{(2)}, \dots, \hat{h}^{(n)}\}$, we define the $[\![\cdots]\!]_{\alpha\beta}$ and $[\![\cdots]\!]$ symbols according to

$$[\![\hat{h}^{(1)}\hat{h}^{(2)}\cdots\hat{h}^{(n)}]\!]_{\alpha\beta} \equiv \hat{h}^{(1)}_{\alpha\mu_1}\,\eta^{\mu_1\mu_2}\,\hat{h}^{(2)}_{\mu_1\mu_2}\,\eta^{\mu_2\mu_3}\,\cdots\,\hat{h}^{(n)}_{\mu_n\beta} \tag{3.1}$$

$$[\![\hat{h}^{(1)}\hat{h}^{(2)}\cdots\hat{h}^{(n)}]\!] \equiv [\![\eta^{\alpha\beta}\,\hat{h}^{(1)}\hat{h}^{(2)}\cdots\hat{h}^{(n)}]\!]_{\alpha\beta} \tag{3.2}$$

such that, for example, $[\![1]\!]_{\alpha\beta} = \eta_{\alpha\beta}$ and $[\![1]\!] = 4$.

This chapter introduces important definitions and conventions from general relativity, as well as introducing the Randall-Sundrum 1 (RS1) model which is the primary topic of this dissertation. It also introduces several original results related to the RS1 Lagrangian. This includes an updated 5D weak field expanded (WFE) RS1 Lagrangian, which we originally published in Appendix A of [18] using a different form of the Einstein-Hilbert Lagrangian. We also demonstrate for the first time that all terms in the 5D WFE RS1 Lagrangian which are proportional to $(\partial_\varphi^2|\varphi|)$ and $(\partial_\varphi|\varphi|)$ can be repackaged into a physically-irrelevant total derivative.

- Section 3.2 establishes our tensor conventions, including the covariant derivative, Riemann curvature, Ricci scalar, and Einstein-Hilbert Lagrangian; rewrites the Einstein-Hilbert Lagrangian into a more convenient form; and derives the extra-dimensional graviton resulting from the Einstein-Hilbert Lagrangian.

- Section 3.3 motivates the construction of the Randall-Sundrum 1 background metric by considering what modifications are required in order to accommodate the nonzero extrinsic curvature it necessarily implies at its branes. The background metric is then perturbed to generate the full 5D RS1 model, with a metric that depends on 5D fields $\hat{h}_{\mu\nu}(x,y)$ and $\hat{r}(x)$. The final subsection demonstrates that terms proportional to $(\partial_\varphi^2|\varphi|)$ and $(\partial_\varphi|\varphi|)$ combine to form physically-irrelevant total derivatives, and then introduces a new term to the 5D RS1 model Lagrangian to automate the removal of such terms.

- Section 3.4 weak field expands the 5D RS1 model Lagrangian as a series in the 5D fields $\hat{h}_{\mu\nu}(x,y)$ and $\hat{r}(x)$ to second order in the coupling, $\mathcal{O}(\kappa_{5D}^2)$. This 5D weak field expanded (WFE) RS1 Lagrangian is the principle result of this chapter, and updates the expressions we originally published in Appendix A of [18].

- Section 3.5 is an appendix which details certain formula used in the weak field expansion procedure.

3.2 Motivations, Definitions, and Conventions

3.2.1 Revisiting the Metric

In the previous chapter, we explored the consequences of demanding that the speed of light be globally conserved between inertial reference frames in flat 4D spacetime, i.e. that every finite spacetime interval that is light-like according to one observer is also light-like to all other observers. This led us to the Poincaré group and eventually the characterization of external particles on that spacetime. This chapter generalizes those assumptions: we consider a metric G on X-dimensional spacetime that is a function of X-dimensional spacetime coordinates x, where $X \geq 4$. We label the first four coordinates in the usual way, e.g. when $X = 4$ indices label coordinates according to $x^\mu \equiv (x^0, x^1, x^2, x^3)$, with x^0 denoting a time coordinate. Subsequent coordinate indices will start from 5, e.g. when $X = 5$ we have $x^M \equiv (x^0, x^1, x^2, x^3, x^5)$.

As before, each observer is associated with a specific choice of coordinates, and via those coordinates the observer can write the metric G in terms of components G_{MN}. By assumption, the tensor G_{MN} is symmetric and nondegenerate. We use the "mostly-minus" convention, which establishes that in any particular coordinate patch G_{MN} has a single positive eigenvalue among otherwise negative eigenvalues. A vector v^M is time-like (space-like) if it possesses a positive (negative) inner product with itself with respect to the metric: $G_{MN}v^M v^N > 0$ ($G_{MN}v^M v^N < 0$). If instead this inner product vanishes, $G_{MN}v^M v^N = 0$, then v^M is declared light-like.

The metric G defines the invariant spacetime interval ds^2 across any infinitesimal displacement dx^M:

$$ds^2 = G_{MN} dx^M dx^N \tag{3.3}$$

where the matrix $[G_{MN}]$ is invertible. We write the inverse matrix as $[\tilde{G}^{MN}]$, the components of which by definition satisfy $\tilde{G}^{MN} G_{NP} = \mathbb{1}^M_P$. By using a tilde to denote matrix inverses, we save space, reduce notational clutter, and prevent potential confusion later.

In the last chapter, the metric G_{MN} was assumed to equal $\eta_{\mu\nu}$ and we only considered linear transformations that mapped $\eta_{\mu\nu}$ to itself. We now relax those requirements: G_{MN} can be a nontrivial function of the coordinates x^M, and we consider (possibly nonlinear) coordinate transformations that map x^M to new coordinates $\overline{x}^{\overline{M}}$ which thereby map G_{MN} to a new form $\overline{G}_{\overline{MN}}$. This is the topic of the next subsection.

3.2.2 Diffeomorphisms, Tensors

A diffeomorphism is a transformation that maps the coordinates of one reference frame to the coordinates of another reference frame. In order to locally preserve the speed of light between any two reference frames, we demand ds^2 be invariant under diffeomorphisms. This implies how G must transform. Specifically, if G_{MN} describes spacetime in coordinates x^M and $\overline{G}_{\overline{MN}}$ describes spacetime in coordinates $\overline{x}^{\overline{M}}$, then the infinitesimal displacements at an equivalent point in either description are related according to

$$d\overline{x}^{\overline{M}} = \overline{\mathfrak{D}}^{\overline{M}}{}_{M} dx^M \qquad \text{where} \qquad \overline{\mathfrak{D}}^{\overline{M}}{}_{M} \equiv \left(\frac{\partial \overline{x}^{\overline{M}}}{\partial x^M}\right) dx^M \tag{3.4}$$

We can similarly convert the dx^M on the RHS of this expression to $d\overline{x}^{\overline{M}}$, and thereby we obtain

$$d\overline{x}^{\overline{M}} = \overline{\mathfrak{D}}^{\overline{M}}{}_{N} \mathfrak{D}^{M}{}_{\overline{N}} dx^{\overline{N}} \qquad \text{where} \qquad \mathfrak{D}^{M}{}_{\overline{M}} \equiv \left(\frac{\partial x^M}{\partial \overline{x}^{\overline{M}}}\right) \tag{3.5}$$

which implies, recalling that we use tildes to denote inverses,

$$\overline{\mathfrak{D}}^{\overline{M}}{}_{M} \mathfrak{D}^{M}{}_{\overline{N}} = \mathbb{1}^{\overline{M}}_{\overline{N}} \qquad \text{such that} \qquad \tilde{\mathfrak{D}}^{\overline{M}}{}_{M} = \overline{\mathfrak{D}}^{\overline{M}}{}_{M} \tag{3.6}$$

The requirement that a coordinate transformation leaves the invariant spacetime interval unchanged, i.e.

$$G_{MN}\, dx^M dx^N = ds^2 = \overline{G}_{\overline{MN}}\, d\overline{x}^{\overline{M}} d\overline{x}^{\overline{N}} \tag{3.7}$$

implies that the metric transforms according to

$$\overline{G}_{\overline{MN}} = \mathfrak{D}^{M}{}_{\overline{M}}\, \mathfrak{D}^{N}{}_{\overline{N}}\, G_{MN} \tag{3.8}$$

From this, we can derive the transformation properties of other spacetime tensors.

By definition, any object that transforms like dx^M under a diffeomorphism is called a vector, i.e. v is a vector if

$$\overline{v}^{\overline{M}} = \overline{\mathfrak{D}}^{\overline{M}}{}_{M} v^M \tag{3.9}$$

and is said to have a contravariant index. The vector transformation rule in combination with Eq. (3.8) implies that the covector $(Gv)_M \equiv (G_{MN} v^N)$ corresponding to the vector v^M must transform under diffeomorphisms according to

$$\overline{(Gv)}_{\overline{M}} = (\overline{G}_{\overline{MN}} \overline{v}^{\overline{N}}) = \mathfrak{D}^{M}{}_{\overline{M}}\, \mathfrak{D}^{N}{}_{\overline{N}}\, G_{MN}\, \overline{\mathfrak{D}}^{\overline{M}}{}_{N}\, v^N = \mathfrak{D}^{M}{}_{\overline{M}}\, (Gv)_M \tag{3.10}$$

and is said to have a covariant index. More generally, any index that transforms via $\overline{\mathfrak{D}}$ ($\mathfrak{D}$) is termed contravariant (covariant), and an object having m contravariant and n covariant indices is called a rank-(m, n) tensor. A tensor is said to transform covariantly under diffeomorphisms. By contracting all contravariant indices with covariant indices and evaluating all fields at equivalent spacetime points, we guarantee the construction of a diffeomorphism invariant quantity. For example, the inner product $G_{MN} v^M w^N$ of any tangent space vector fields v and w at a spacetime point x is diffeomorphism invariant.

In the gravity literature, the covector $(Gv)_M$ is commonly defined via the symbol v_M. This is a specific instance of a more general rule wherein indices are lowered via the metric G and raised via its inverse $\tilde{G}$. This rule is quite convenient because allows us to immediately know how an index transforms based on whether it is written as a superscript or a subscript. Unfortunately, this convention is not particularly useful for the goals of this dissertation. As demonstrated in this chapter and the next, the metric (when perturbed relative to a background solution) contains particle content, and allowing the metric to be buried in raising and lowering indices will obscure where instances of various fields occur. Therefore, we avoid absorbing the metric into tensors by instead raising or lowering indices via a flat metric $[\eta_{MN}] \equiv \text{Diag}(+1, -1, \cdots, -1)$, which is a popular convention in the weak field expansion literature (more on weak field expansions later in this chapter). Therefore, given a vector v, we define $v_M \equiv (\eta v)_M = \eta_{MN} v^N$. Although the index M in $(Gv)_M$ is covariant, the index M in $v_M = (\eta v)_M$ is still contravariant:

$$\overline{(Gv)}_{\overline{M}} = \mathfrak{D}^M{}_{\overline{M}} (Gv)_M \qquad \text{versus} \qquad \overline{v}_{\overline{M}} = \eta_{\overline{MN}}\, \overline{v}^{\overline{N}} = \overline{\mathfrak{D}}_{\overline{M}N} v^N \tag{3.11}$$

where we assume η is coordinate-independent: $[\overline{\eta}_{\overline{MN}}] = [\eta_{MN}]$.

When constructing a Lagrangian theory of gravity, a diffeomorphism-invariant integration element is vital for defining spacetime integrals. To begin, consider the typical volume element $d^X x$. This is not invariant under the coordinate transformation $x \to \overline{x} = \overline{\mathfrak{D}} x$, and instead

$$d^X \overline{x} = \left|\det \overline{\mathfrak{D}}\right|\, d^X x \tag{3.12}$$

where $\det \overline{\mathfrak{D}} \equiv \det[\overline{\mathfrak{D}}^{\overline{M}}{}_M]$. Our goal is to combine this with other objects as to create a diffeomorphism-invariant measure. Thankfully, we immediately have access to another object that transforms proportional to factors of $|\det \overline{\mathfrak{D}}|$: by taking the determinant of the transformation rule of the metric Eq. (3.8), we find that $|\det G|$ and $|\det \overline{G}|$ are related according to

$$|\det \overline{G}| = |\det \mathfrak{D}|^2\, |\det G| = \frac{|\det G|}{\left|\det \overline{\mathfrak{D}}\right|^2} \tag{3.13}$$

where we have used that $\overline{\mathfrak{D}} = \tilde{\mathfrak{D}}$, such that

$$\sqrt{|\det \overline{G}|} = \frac{\sqrt{|\det G|}}{\left|\det \overline{\mathfrak{D}}\right|} \tag{3.14}$$

Combining Eqs. (3.12) and (3.14), we find that $\sqrt{|\det G|}\, d^X x$ is diffeomorphism invariant:

$$\sqrt{|\det \overline{G}|}\, d^X \overline{x} = \frac{\sqrt{|\det G|}}{\left|\det \overline{\mathfrak{D}}\right|} \left|\det \overline{\mathfrak{D}}\right|\, d^X x = \sqrt{|\det G|}\, d^X x \tag{3.15}$$

This is the invariant (spacetime) volume element we desired. Because we use the mostly-minus convention, $\text{sign}(\det G) = (-1)^{X-1}$, such that $\sqrt{|\det G|} = \sqrt{\mp \det G}$ if X is even or odd respectively. For succinctness, we define $\sqrt{G} \equiv \sqrt{|\det G|}$. If $\phi(x)$ is a diffeomorphism invariant scalar field, then $\int d^X x \sqrt{G}\, \phi(x)$ is diffeomorphism invariant as well, such that we can construct a coordinate-independent action.

On occasion, it will be useful to purposefully symmetrize (antisymmetrize) some collection of indices, which we denote with parentheses (brackets). For example,

$$T_{(a_1 \cdots a_\ell)} \equiv \frac{1}{\ell!} \sum_\pi T_{a_{\pi(1)} \cdot a_{\pi(\ell)}} \qquad T_{[a_1 \cdots a_\ell]} \equiv \frac{1}{\ell!} \sum_\pi \text{sign}(\pi)\, T_{a_{\pi(1)} \cdot a_{\pi(\ell)}} \tag{3.16}$$

where $\text{sign}(\pi) = \pm 1$ if the permutation π is even (odd). Sometimes symmetrization (antisymmetrization) will occur for indices across multiple tensors; in any case, the indices contained between the parentheses (brackets) are included in the procedure.

3.2.3 Covariant Derivative, Christoffel Symbol, Lie Derivative

Beyond any specific coordinate-dependent effects, the metric encodes curvature inherent to spacetime. This curvature implies that the usual coordinate derivative $\partial_M \equiv (\partial/\partial x^M)$ is not necessarily a natural derivative on spacetime, e.g. although ∂_M dictates translations in the coordinate x^M, information about vectors or covectors is not necessarily translated in a coordinate-covariant way. Furthermore, although the index M of $\partial_M \phi$ (where ϕ is a generic spacetime scalar field) is covariant under diffeomorphisms,

$$\partial_M \phi \equiv \frac{\partial \phi}{\partial x^M} \qquad \mapsto \qquad \overline{\partial}_{\overline{M}} \overline{\phi} \equiv \frac{\partial \overline{\phi}}{\partial \overline{x}^{\overline{M}}} = \mathfrak{D}^M{}_{\overline{M}} (\partial_M \phi) \tag{3.17}$$

the equivalent index on the derivative of a more complicated tensor such as $\partial_M v^N$ (where v is a generic spacetime vector field) is not diffeomorphism covariant,

$$\partial_M v^N \equiv \frac{\partial v^N}{\partial x^M} \qquad \mapsto \qquad \overline{\partial}_{\overline{M}} v^{\overline{N}} = \mathfrak{D}^M{}_{\overline{M}}\, \overline{\partial}_M [\overline{\mathfrak{D}}^{\overline{N}}{}_N v^N] \neq \mathfrak{D}^M{}_{\overline{M}}\, \overline{\mathfrak{D}}^{\overline{N}}{}_N\, \overline{\partial}_M v^N \tag{3.18}$$

which presents an obstacle when constructing a diffeomorphism-invariant action. To address these problems, we require a derivative that incorporates the structure of spacetime.

Two derivatives of this sort commonly occur in general relativity calculations: the covariant derivative and the Lie derivative. Both are derivatives in the traditional sense—i.e. they are linear maps which obey the Leibniz rule—although they differ in their details and applications. The covariant derivative is particularly useful when constructing Lagrangians on curved spacetime, depends on the metric G, and transforms a rank-(m, n) tensor into a rank-$(m, n+1)$ tensor. In contrast, the Lie derivative generalizes the directional derivative of flat spacetime, is independent of the metric G, and transforms a rank-(m, n) tensor into another rank-(m, n) tensor.

For the covariant derivative, we utilize what is called the Levi-Civita connection ∇_A, which is the unique affine connection that is simultaneously compatible with the metric $(\nabla_A G_{MN} = 0)$ and torsion-free. Its action on a given tensor depends on the rank of that tensor, e.g. for a scalar field $\phi(x)$ the covariant derivative reduces to the usual derivative,

$$\nabla_A \phi = \partial_A \phi \tag{3.19}$$

whereas for a vector $v^M(X)$,

$$\nabla_A v^M = \partial_A v^M + \Gamma^M_{AN} v^N \tag{3.20}$$

where Γ^P_{MN} is the Christoffel symbol,

$$\Gamma^P_{MN} \equiv \frac{1}{2}\tilde{G}^{PQ}(\partial_M G_{NQ} + \partial_N G_{MQ} - \partial_Q G_{MN}) \tag{3.21}$$

Note that the Christoffel symbol is symmetric in its lower indices, i.e. $\Gamma^P_{MN} = \Gamma^P_{NM}$. Despite its suggestive index structure, the Christoffel symbol does not transform like a spacetime tensor (indeed, it cannot because $(\partial_A v^M)$ is not a spacetime tensor but $\nabla_A v^M$ is). Taking the covariant derivative of a tensor possessing multiple contravariant indices proceeds similarly, with the addition of the appropriate number of terms each containing a Christoffel symbol contracted with a different index. When covariant indices are present, the Chistoffel symbol terms are instead subtracted, e.g.

$$\nabla_A v_M = \partial_A v_M - \Gamma^N_{AM} v_N \tag{3.22}$$

Multiple covariant indices generalize accordingly via the additional subtraction of a Christoffel symbol-containing term per covariant index. Combining the contravariant and covariant behaviours yields the formula for a generic rank-(m, n) tensor. Because of its compatibility with the metric, the covariant derivative of any function of the metric alone vanishes.

The Lie derivative is a coordinate-invariant measure of the change in a spacetime tensor with respect to a vector field. It is the generalization of the standard directional derivative in flat spacetimes. Like the covariant derivative, its exact operation depends on the rank of the tensor it operates on. For example, given a vector field v^M and scalar ϕ, the appropriate relation is

$$\pounds_v \phi \equiv (v \cdot \partial)\phi \tag{3.23}$$

whereas given a vector field w^M it is

$$\pounds_v w^M \equiv (v \cdot \partial) w^M - (\partial_N v^M) w^N \tag{3.24}$$

and given a covector field w_M it is

$$\pounds_v w_M \equiv (v \cdot \partial) w_M + (\partial_M v^N) w_N \tag{3.25}$$

where $v \cdot \partial \equiv v^M \partial_M$. These equations also hold true if the derivatives ∂_A are replaced with covariant derivatives ∇_A. We will utilize the Lie derivative when we calculate extrinsic curvature in the RS1 model.

3.2.4 Curvature

The metric expressed in a given coordinate system enables a quantitative measure of the curvature of spacetime. For example, the Riemann curvature tensor measures spacetime curvature via the failure of covariant derivatives to commute when acting on a generic covector:

$$R_{ABC}{}^{D} w_D \equiv (\nabla_A \nabla_B - \nabla_B \nabla_A) w_C \tag{3.26}$$

By replacing the covariant derivatives with their expression in terms of Christoffel symbols, we attain a formula for the Riemann curvature that will prove more useful for our computations:

$$R_{ABC}{}^{D} \equiv (\partial_B \Gamma^D_{AC}) - (\partial_A \Gamma^D_{BC}) + \Gamma^E_{AC}\Gamma^D_{BE} - \Gamma^E_{BC}\Gamma^D_{AE} \tag{3.27}$$
$$= (\partial_{[B}\Gamma^D_{A]C}) + \Gamma^E_{C[A}\Gamma^D_{B]E} \tag{3.28}$$

Whether or not an additional minus sign is included in the above definition amounts to a convention; across the literature, both choices are used with nearly equal frequency and without much consistency across in any given subfield. Consequently, ambiguity in this convention can be a source of many headaches. The Riemann curvature tensor is frequently self-contracted to form the Ricci tensor,

$$R_{AC} \equiv R_{ABC}{}^{B} \tag{3.29}$$

or subsequently contracted with the inverse metric to form the Ricci scalar (scalar curvature)

$$R \equiv \tilde{G}^{AC} R_{AC} \tag{3.30}$$

The Ricci scalar is an important constituent of the Einstein-Hilbert Lagrangian, which in turn is a foundational contribution to the gravity Lagrangians discussed in the next section.

3.2.5 Einstein-Hilbert Lagrangian, Cosmological Constant, Einstein Field Equations

The Einstein-Hilbert action S_{EH} and the Einstein-Hilbert Lagrangian $\mathcal{L}_{\text{EH}}$ are defined according to

$$S_{\text{EH}} \equiv -\frac{2}{\kappa^2_{\text{XD}}} \int d^X x \ \sqrt{G}\, R \equiv -\frac{2}{\kappa^2_{\text{XD}}} \int d^X x \ \mathcal{L}_{\text{EH}} \tag{3.31}$$

where $\sqrt{G} \equiv \sqrt{|\det G|}$. The negative prefactor $(-2/\kappa^2_{\text{XD}})$ is directly tied to the sign of the Riemann curvature which we chose in the previous section, and ensures properly normalized (positive energy) graviton modes. To derive the equations of motion for the metric, consider varying the Einstein-Hilbert action with respect to the inverse metric $\tilde{G}^{AB}$. Because

$$\frac{\delta}{\delta \tilde{G}^{AB}} \left[\sqrt{G}\right] = -\frac{1}{2}\sqrt{G}\, G_{AB} \qquad\qquad \frac{\delta}{\delta \tilde{G}^{AB}} [R] = R_{AB} \tag{3.32}$$

the first variation of S_{EH} yields, assuming vanishing surface terms,

$$\delta S_{\text{EH}} = -\frac{2}{\kappa^2_{\text{XD}}} \int d^X x \ \sqrt{G} \left[R_{AB} - \frac{1}{2} G_{AB} R\right] \delta \tilde{G}^{AB} \tag{3.33}$$

such that, without additional modifications, the equations of motion equal

$$\mathcal{G}_{AB} \equiv R_{AB} - \frac{1}{2} G_{AB} R = 0 \tag{3.34}$$

where $\mathcal{G}_{AB}$ is the Einstein tensor.

There are two other Lagrangians commonly added to $\mathcal{L}_{\text{EH}}$. The first we consider is the cosmological constant Lagrangian,

$$\mathcal{L}_{\text{CC}} \equiv -\frac{4}{\kappa_{\text{XD}}^2} \sqrt{G}\, \Lambda \tag{3.35}$$

where Λ is a real number. The variation of $\mathcal{L}_{\text{CC}}$ yields

$$\frac{\delta}{\delta \tilde{G}^{AB}} \left[\mathcal{L}_{\text{CC}}\right] = -\frac{2}{\kappa_{\text{XD}}^2} \sqrt{G}\, (-\Lambda\, G_{AB}) \tag{3.36}$$

The second we consider is the matter Lagrangian, the form of which is left mostly ambiguous unless applied to a specific choice of matter fields. Its contribution is typically written with a factor of the invariant volume element already accounted for but (in contrast to the previous two Lagrangian contributions considered) without any factors of κ_{XD}, as $\sqrt{G}\, \mathcal{L}_{\text{M}}$. Its variation with respect to the inverse metric equals

$$\frac{\delta}{\delta \tilde{G}^{AB}} \left[\sqrt{G}\, \mathcal{L}_{\text{M}}\right] = \frac{1}{2} \sqrt{G}\, \mathcal{T}_{AB} \tag{3.37}$$

where $\mathcal{T}_{AB}$ is the stress-energy tensor

$$\mathcal{T}_{AB} \equiv 2 \frac{\delta \mathcal{L}_{\text{M}}}{\delta \tilde{G}^{AB}} - G_{AB} \mathcal{L}_{\text{M}} \tag{3.38}$$

which expresses the stress-energy content generated by the matter fields.

Therefore, for the Lagrangian,

$$\mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{CC}} + \sqrt{G}\, \mathcal{L}_{\text{M}} \tag{3.39}$$

the equations of motion equal

$$\mathcal{G}_{AB} - \Lambda G_{AB} = \frac{\kappa_{\text{XD}}^2}{4} \mathcal{T}_{AB} \tag{3.40}$$

These gravitational equations of motion (and extensions thereof) are the Einstein field equations, and imply that a cosmological constant permeating all of spacetime and/or the presence of matter causes the curving of spacetime through the Einstein tensor $\mathcal{G}_{AB}$. The curvature of spacetime is closely tied to the presence of fields on that spacetime, not unlike the close ties between electric fields and electric charges.

The aforementioned Lagrangians describe bulk gravitational physics; when it becomes necessary, we will extend these to incorporate spacetime matter and/or energy localized to submanifolds, such as branes.

To conclude this section, we note that the Einstein-Hilbert Lagrangian can be rewritten using integration-by-parts into a form wherein any given instance of the metric is never differentiated more than once:

$$\mathcal{L}_{\text{EH}} \cong -\frac{2}{\kappa^2_{\text{XD}}}\sqrt{G}\tilde{G}^{MN}\left[\Gamma^{Q}_{NP}\Gamma^{P}_{MQ} - \Gamma^{P}_{QP}\Gamma^{Q}_{MN}\right] \tag{3.41}$$

The symbol $\cong$ denotes equality as an integrand of the action via integration by parts. This alternate form is derived in next subsection.

3.2.6 Rewriting the Einstein-Hilbert Lagrangian

The Einstein-Hilbert Lagrangian is defined, traditionally, in terms of the scalar curvature as

$$\mathcal{L}_{\text{EH}} = -\frac{2}{\kappa^2_{\text{XD}}}\int d^X x\ \sqrt{G}\,R \tag{3.42}$$

$$= -\frac{2}{\kappa^2_{\text{XD}}}\int d^X x\ \sqrt{G}\,\tilde{G}^{MN}\left[(\partial_P\Gamma^{P}_{MN}) - (\partial_M\Gamma^{P}_{PN}) + \Gamma^{Q}_{MN}\Gamma^{P}_{PQ} - \Gamma^{Q}_{MP}\Gamma^{P}_{NQ}\right] \tag{3.43}$$

However, we find it more useful to work with an alternate form of $\mathcal{L}_{\text{EH}}$ which is attained through integration by parts. Integration by parts will move the derivatives acting on Christoffel symbols in the first two terms of Eq. (3.43) onto $\sqrt{G}\,\tilde{G}^{MN}$, such that all Christoffel symbols are no longer differentiated. This will also eliminate all twice-differentiated quantities from the Einstein-Hilbert Lagrangian.

In order to eventually simplify the expressions we obtain from this integration by parts procedure, recall that any function which only depends on the metric has vanishing covariant derivative. Therefore,

$$0 = \nabla_C\tilde{G}^{MN} = (\partial_C\tilde{G}^{MN}) + \Gamma^{M}_{AC}\tilde{G}^{AN} + \Gamma^{N}_{AC}\tilde{G}^{MA} \tag{3.44}$$

such that

$$(\partial_C\tilde{G}^{MN}) = -\tilde{G}^{AN}\Gamma^{M}_{AC} - \tilde{G}^{MA}\Gamma^{N}_{AC} \tag{3.45}$$

and[1]

$$0 = \nabla_C\sqrt{G} = (\partial_C\sqrt{G}) - \sqrt{G}\,\Gamma^{A}_{AC} \tag{3.46}$$

such that

$$(\partial_C\sqrt{G}) = \sqrt{G}\,\Gamma^{A}_{AC} \tag{3.47}$$

[1]That $\sqrt{G} = \sqrt{\det G}$ requires a nontrivial covariant derivative arises from the fact that $\det G$ transforms nontrivially under diffeomorphisms, as originally mentioned in Eq. (3.14). In particular, $\sqrt{G}$ is a scalar density with unit weight, where weight refers to the constant multiplying $-\sqrt{G}\,\Gamma^{A}_{AC}$ in Eq. (3.46). For example, $\det G$ has weight $+2$, and thus its covariant derivative contains instead the term $-2\sqrt{G}\,\Gamma^{A}_{AC}$.

Together these results imply that

$$\partial_C \left(\sqrt{G}\,\tilde{G}^{MN}\right) = (\partial_C\sqrt{G})\,\tilde{G}^{MN} + \sqrt{G}\,(\partial_C\tilde{G}^{MN}) \tag{3.48}$$

$$= \sqrt{G}\left[\tilde{G}^{MN}\,\Gamma^A_{AC} - \tilde{G}^{AN}\Gamma^M_{AC} - \tilde{G}^{MA}\Gamma^N_{AC}\right] \tag{3.49}$$

and we are now ready to begin rewriting the Einstein-Hilbert Lagrangian.

Consider the first term of Eq. (3.43). It is proportional to

$$+\sqrt{G}\,\tilde{G}^{MN}\,(\partial_P\Gamma^P_{MN}) \cong -\partial_P\left[\sqrt{G}\,\tilde{G}^{MN}\right]\,\Gamma^P_{MN} \tag{3.50}$$

$$= \sqrt{G}\left[-\tilde{G}^{MN}\,\Gamma^A_{AP}\Gamma^P_{MN} + \tilde{G}^{AN}\Gamma^M_{AP}\Gamma^P_{MN} + \tilde{G}^{MA}\Gamma^N_{AP}\Gamma^P_{MN}\right] \tag{3.51}$$

$$= \sqrt{G}\,\tilde{G}^{MN}\left[-\Gamma^Q_{MN}\,\Gamma^P_{PQ} + 2\,\Gamma^Q_{MP}\Gamma^P_{NQ}\right] \tag{3.52}$$

where integration by parts was used in the first line, and the last line utilizes both index relabeling and the index symmetries of $\tilde{G}^{MN}$ and Γ^P_{MN}. Similarly, the second term of Eq. (3.43) is proportional to

$$-\sqrt{G}\,\tilde{G}^{MN}\,(\partial_M\Gamma^P_{PN}) \cong +\partial_M\left[\sqrt{G}\,\tilde{G}^{MN}\right]\,\Gamma^P_{PN} \tag{3.53}$$

$$= \sqrt{G}\left[+\tilde{G}^{MN}\,\Gamma^A_{AM}\Gamma^P_{PN} - \tilde{G}^{AN}\Gamma^M_{AM}\Gamma^P_{PN} - \tilde{G}^{MA}\Gamma^N_{AM}\Gamma^P_{PN}\right] \tag{3.54}$$

$$= \sqrt{G}\,\tilde{G}^{MN}\left[-\Gamma^Q_{MN}\Gamma^P_{PQ}\right] \tag{3.55}$$

Substituting these results into Eq. (3.43) yields the desired alternate form of the Einstein-Hilbert Lagrangian:

$$\mathcal{L}_{\text{EH}} = -\frac{2}{\kappa^2_{\text{XD}}}\int d^Xx\;\sqrt{G}\,\tilde{G}^{MN}\left[\Gamma^Q_{MP}\Gamma^P_{NQ} - \Gamma^Q_{MN}\Gamma^P_{PQ}\right] \tag{3.56}$$

Because each Christoffel symbol contains exactly one derivative per term by definition, $\mathcal{L}_{\text{EH}}$ contains exactly two derivatives per term. One advantage of this alternate form (which lacks the $\partial\Gamma \supset \partial\partial G$ terms of the traditional form) is that it ensures those two derivatives are never applied to the same object in any given term.

3.2.7 Deriving the Graviton

Consider the aforementioned D-dimensional gravitational Lagrangian in the absence of a cosmological constant and matter, so that the relevant Lagrangian is exclusively the Einstein-Hilbert Lagrangian Eq. (3.56). Furthermore, suppose we interpret the given metric G_{MN} as only slightly perturbed away from the (flat) background metric $\eta_{MN} \equiv \text{Diag}(+1, -1, \ldots, -1)$, e.g. $G_{MN} \equiv \eta_{MN} + \kappa_{\text{XD}}\hat{H}_{MN}$ for some spacetime-dependent perturbation $\hat{H}_{MN}$. This enables us to calculate $\mathcal{L}_{\text{EH}}$ as a perturbative series in $\hat{H}$. In general, the process of expanding a metric about a background metric that solves the Einstein field equations is called weak field expansion (WFE). At present, we will weak field expand the Einstein-Hilbert Lagrangian through $\mathcal{O}(\hat{H}^2)$.

First, note that weak field expansion of the Christoffel symbol corresponding to the G_{MN} described above yields

$$\Gamma^P_{MN} \equiv \frac{1}{2}\tilde{G}^{PQ}(\partial_M G_{NQ} + \partial_N G_{MQ} - \partial_Q G_{MN}) \tag{3.57}$$

$$= \frac{\kappa_{\text{XD}}}{2}\left[\sum_{n=0}^{+\infty}(-1)^n [\![(\kappa_{\text{XD}}\hat{H})^n]\!]^{PQ}\right]\left[(\partial_M \hat{H}_{NQ}) + (\partial_N \hat{H}_{MQ}) - (\partial_Q \hat{H}_{MN})\right] \tag{3.58}$$

$$= \frac{\kappa_{\text{XD}}}{2}\left[(\partial_M \hat{H}^P_N) + (\partial_N \hat{H}^P_M) - (\partial^P \hat{H}_{MN})\right] + \mathcal{O}(\hat{H}^2) \tag{3.59}$$

where we utilize the twice-squared bracket notation introduced in Chapter 2.2.1. We need only expand the Christoffel symbols to first order in the field $\hat{H}$ to obtain an overall $\mathcal{O}(\hat{H}^2)$ result because they begin at that order and $\mathcal{L}_{\text{EH}}$ is composed of products of pairs of Christoffel symbols.

When these expansions are substituted into the Einstein-Hilbert Lagrangian, we find

$$\mathcal{L}_{\text{EH}} \cong -\frac{2}{\kappa^2_{\text{XD}}}\eta^{MN}\left[\Gamma^Q_{NP}\Gamma^P_{MQ} - \Gamma^P_{QP}\Gamma^Q_{MN}\right] + \mathcal{O}(H^3) \tag{3.60}$$

$$= (\partial^A \hat{H}_{AB})(\partial^B \hat{H}) - (\partial_A \hat{H}_{BC})(\partial^C \hat{H}^{AB}) + \frac{1}{2}(\partial_A \hat{H}_{BC})^2 - \frac{1}{2}(\partial_B \hat{H})^2 + \mathcal{O}(\hat{H}^3) \tag{3.61}$$

where the $\sqrt{G}\,\tilde{G}^{MN}$ prefactor has already been expanded in the first line (more information about the weak field expansion of $\sqrt{G}$ and $\tilde{G}^{MN}$ can be found in Section 3.5). When $X = 4$, Eq. (3.61) is precisely the massless spin-2 Lagrangian from Section 2.8. When $X \neq 4$, the equations of motion still go through as-is and constrain the propagation of $\hat{H}_{MN}$ such that the field must be transverse and traceless: $(\partial^M \hat{H}_{MN}) = \hat{H}^M_M = 0$. In general, after applying the equations of motion, an X-dimensional graviton has $(X+1)X/2 - 2X = (X-3)X/2$ degrees of freedom. Therefore, a 4D graviton has 2 degrees of freedom, whereas a 5D graviton has 5 degrees of freedom.

Consider the effect of a coordinate transformation $x \to \overline{x} = \overline{\mathfrak{D}}x$ on the field $\hat{H}_{MN}$, as transmitted through the known transformation properties of G_{MN}. In particular, suppose the diffeomorphism is of the form of a coordinate-dependent spacetime translation $\overline{x}^M = x^M + \epsilon^M(x)$ for some vector field ϵ^M, and that the vector components ϵ^M are at most comparable in magnitude to the field components $\hat{H}_{MN}$ so that we may simultaneously expand in ϵ, e.g. $\mathcal{O}(\epsilon) \sim \mathcal{O}(\hat{H})$. We now demonstrate that this spacetime translation exactly reproduces the gauge freedom of the massless spin-2 Lagrangian when $X = 4$.

The aforementioned diffeomorphism implies $\overline{\mathfrak{D}}^{\overline{M}}{}_M = (\partial\overline{x}^{\overline{M}}/\partial x^M) = \mathbb{1}^{\overline{M}}_M + (\partial_M \epsilon^{\overline{M}})$, so that general coordinate invariance demands

$$G_{MN} = \overline{\mathfrak{D}}^{\overline{M}}{}_{M}\,\overline{\mathfrak{D}}^{\overline{N}}{}_{N}\,\overline{G}_{\overline{MN}} \tag{3.62}$$

$$= \left[\mathbb{1}^{\overline{M}}_{M} + (\partial_M \epsilon^{\overline{M}})\right]\left[\mathbb{1}^{\overline{N}}_{N} + (\partial_N \epsilon^{\overline{N}})\right]\overline{G}_{\overline{MN}} \tag{3.63}$$

$$= \overline{G}_{MN} + (\partial_M \epsilon^{\overline{M}})\overline{G}_{\overline{M}N} + (\partial_N \epsilon^{\overline{N}})\overline{G}_{M\overline{N}} + (\partial_M \epsilon^{\overline{M}})(\partial_N \epsilon^{\overline{N}})\overline{G}_{\overline{MN}} \tag{3.64}$$

which is an exact result. To proceed further, series expand the quantity $\overline{G}(\overline{x}) = \overline{G}(x+\epsilon)$ in ϵ through $\mathcal{O}(\epsilon)$:

$$\overline{G}_{MN}(\overline{x}) = \overline{G}_{MN}(x) + \epsilon^{\overline{M}}\partial_{\overline{M}}\overline{G}_{MN}(x) + \mathcal{O}(\epsilon^2) \tag{3.65}$$

such that,

$$G_{MN} = \overline{G}_{MN} + (\epsilon\cdot\partial)\overline{G}_{MN} + (\partial_M \epsilon^P)\overline{G}_{PN} + (\partial_N \epsilon^P)\overline{G}_{MP} + \mathcal{O}(\epsilon^2) \tag{3.66}$$

where all fields are expressed as functions of the coordinates x. This completes the expansion in ϵ. Note that this can be succinctly expressed in terms of the Lie derivative

$$\pounds_\epsilon \overline{G}_{MN} = (\overline{G}_{MN} - G_{MN}) + \mathcal{O}(\epsilon^2) \tag{3.67}$$

which—given that we performed an infinitesimal coordinate translation—confirms its role as a direction derivative. Next, expand each term in powers of $\hat{H}$, and remember that $\hat{H}$ and ϵ are componentwise comparable in magnitude: per term of Eq. (3.66), we find

$$G_{MN} = \eta_{MN} + \kappa_{\text{XD}}\hat{H}_{MN} \tag{3.68}$$

$$\overline{G}_{MN} = \eta_{MN} + \kappa_{\text{XD}}\hat{\overline{H}}_{MN} \tag{3.69}$$

$$(\epsilon\cdot\partial)\overline{G}_{MN} = (\epsilon\cdot\partial)\hat{\overline{H}}_{MN} \qquad = \mathcal{O}(\epsilon^2, \epsilon\hat{\overline{H}}, \hat{\overline{H}}^2) \tag{3.70}$$

$$(\partial_M \epsilon^P)\overline{G}_{PN} = (\partial_M \epsilon^P)(\eta_{PN} + \hat{\overline{H}}_{PN}) \qquad = (\partial_M \epsilon_N) + \mathcal{O}(\epsilon^2, \epsilon\hat{\overline{H}}, \hat{\overline{H}}^2) \tag{3.71}$$

$$(\partial_N \epsilon^P)\overline{G}_{MP} = (\partial_N \epsilon^P)(\eta_{MP} + \hat{\overline{H}}_{MP}) \qquad = (\partial_N \epsilon_M) + \mathcal{O}(\epsilon^2, \epsilon\hat{\overline{H}}, \hat{\overline{H}}^2) \tag{3.72}$$

such that

$$\kappa_{\text{XD}}\hat{H}_{MN} = \kappa_{\text{XD}}\hat{\overline{H}}_{MN} + (\partial_M \epsilon_N) + (\partial_N \epsilon_M) + \pounds_\epsilon \hat{\overline{H}}_{MN} + \mathcal{O}(\epsilon^2) \tag{3.73}$$

$$= \kappa_{\text{XD}}\hat{\overline{H}}_{MN} + (\partial_M \epsilon_N) + (\partial_N \epsilon_M) + \mathcal{O}(\epsilon^2, \epsilon\hat{\overline{H}}, \hat{\overline{H}}^2) \tag{3.74}$$

This mean that (dropping the distinction between the new and old field labels from here), as far as the field $\hat{H}_{MN}$ is concerned, an infinitesimal coordinate translation corresponds to the field transformation

$$\kappa_{\text{XD}}\hat{H}_{MN} \to \kappa_{\text{XD}}\hat{H}_{MN} + (\partial_M \epsilon_N) + (\partial_N \epsilon_M) + \mathcal{O}(\epsilon^2) \tag{3.75}$$

which is precisely the gauge invariance exhibited by the massless spin-2 Lagrangian when $X = 4$.

3.3 The Randall-Sundrum 1 Model

3.3.1 Deriving The Background Metric

In this subsection, the Randall-Sundrum 1 (RS1) model background metric is motivated and derived. In the next subsection, we perturb this background metric and thereby obtain the full RS1 theory.

As mentioned in the introduction, the RS1 model is a five-dimensional model of gravity with nonfactorizable geometry that was introduced in 1999 in order to solve the hierarchy problem. Relative to the usual four-dimensional spacetime, the RS1 model adds a finite extra dimension of space parameterized by a coordinate y ranging from $y = 0$ to $y = \pi r_c$, where r_c is called the compactification radius. The size πr_c of the extra-dimension is assumed small so that the five-dimensional nature of spacetime remains hidden at low energies. The four-dimensional hypersurfaces defined by $y = 0$ and $y = \pi r_c$ are called branes, and the five-dimensional region between those branes is called the bulk.

The RS1 construction possesses two additional features not mentioned in the previous paragraph: warping of the 4D spacetime relative to the extra dimension and orbifold invariance. Because we will discuss the former property at length later in this section, let us first focus on orbifold invariance. In order that spacetime truly be truncated at the branes, any physically-relevant 5D fields cannot be allowed to oscillate beyond the branes, and thereby their derivatives with respect to y must vanish. This can be ensured by extending the extra dimension so that y covers $[-\pi r_c, +\pi r_c]$ and then demanding that the so-called orbifold reflection $y \to -y$ is a symmetry of the invariant spacetime interval ds^2. Having done this, we can extend y to the entire real line by also declaring the discrete translation $y \to y + 2\pi r_c$ as another symmetry of ds^2. This discrete translational symmetry suggests we can just as well think of the extra dimension as a circle of radius r_c parameterized by an angle $\varphi \equiv y/r_c$, with the discrete translation corresponding to rotating the entire circle about its center by 2π radians. (Despite this extension, we will limit any integrals over the extra dimension to the finite domain $y \in [-\pi r_c, +\pi r_c]$, or equivalently $\varphi \in [-\pi, +\pi]$.) If we imagine this circle to be drawn on a piece of paper, then the identification of points via the orbifold reflection corresponds to folding the paper in half along the line between the points at $\phi = 0$ and $\phi = \pm\pi$ and declaring any points which overlap afterwards to be equivalent. From this perspective, if we once again unfold the paper, then the orbifold reflection transformation swaps points across the folding line, such that the only points unchanged by the transformation are the branes at $\phi = 0$ and $\phi = \pi$. In other words, the two branes are uniquely determined as the orbifold fixed points of the RS1 spacetime.

With descriptions of the RS1 coordinates and spacetime symmetries out of the way, we now aim to find a Lagrangian description of the RS1 background metric, although to do so we must include types of terms we have not yet discussed in this chapter. We begin by searching for a background metric of the form

$$G_{MN} = \begin{pmatrix} a(y)\,\eta_{\mu\nu} & 0 \\ 0 & -1 \end{pmatrix} \tag{3.76}$$

where $a(y)$ is a non-trivial positive real function of the extra-dimensional coordinate y that is consistent with the Einstein field equations. The function $a(y)$ provides the aforementioned warping of 4D spacetime relative to the extra dimension. Eq. (3.76) is intentionally written so that the x^μ coordinates are all treated on equal footing, as would be expected from a 4D Poincaré invariant geometry. If $a(y) = 1$, we recover the flat 5D metric η_{MN}; otherwise, this metric (combined with the orbifold condition) necessarily implies a discontinuity in the curvature at the interval endpoints. As will be detailed in a moment, this introduces Dirac delta terms to the Einstein tensor which provide an obstacle to solving the Einstein field equations. Overcoming this obstacle requires extending the techniques utilized thus far to include brane-localized phenomena.

First, note that G_{MN} as written above only depends on y, so $\partial_\alpha G_{MN} = 0$, whereas $\partial_y G_{MN} = (\partial_y a)\, \delta^\mu_M \delta^\nu_N\, \eta_{\mu\nu}$. Consequently, the only independent non-zero Christoffel symbols equal

$$\Gamma^5_{\mu\nu} = -\frac{1}{2}\tilde{G}^{55}(\partial_y G_{\mu\nu}) = \frac{1}{2}\,(\partial_y a)\,\eta_{\mu\nu} \tag{3.77}$$

$$\Gamma^\rho_{\mu 5} = +\frac{1}{2}\tilde{G}^{\rho\sigma}(\partial_y G_{\mu\sigma}) = \frac{1}{2}\,a^{-1}(\partial_y a)\,\eta^\rho_\mu \tag{3.78}$$

which once again only depend on y. Thus, we may calculate

$$\partial_P \Gamma^P_{MN} = \frac{1}{2}\,(\partial_y^2 a)\,\delta^\mu_M \delta^\nu_N\,\eta_{\mu\nu} \tag{3.79}$$

$$\partial_N \Gamma^P_{MP} = \left[2\,a^{-1}(\partial_y^2 a) - 2\,a^{-2}(\partial_y a)^2\right]\,\delta^5_M \delta^5_N \tag{3.80}$$

$$\Gamma^P_{PQ}\Gamma^Q_{MN} = a^{-1}(\partial_y a)^2\,\delta^\mu_M \delta^\nu_N\,\eta_{\mu\nu} \tag{3.81}$$

$$\Gamma^P_{NQ}\Gamma^Q_{MP} = \frac{1}{2}\,a^{-1}(\partial_y a)^2\,\delta^\mu_M \delta^\nu_N\,\eta_{\mu\nu} + a^{-2}(\partial_y a)^2\,\delta^5_M \delta^5_N \tag{3.82}$$

which collectively yield the Ricci tensor

$$R_{MN} = \begin{pmatrix} \frac{1}{2}\left[(\partial_y^2 a) + a^{-1}(\partial_y a)^2\right]\eta_{\mu\nu} & 0 \\ 0 & -a^{-1}\left[2(\partial_y^2 a) - a^{-1}(\partial_y a)^2\right] \end{pmatrix} \tag{3.83}$$

and the scalar curvature

$$R = 4a^{-1}(\partial_y^2 a) + a^{-2}(\partial_y a)^2 \tag{3.84}$$

This allows us to calculate the Einstein tensor, which equals

$$\mathcal{G}_{AB} = R_{AB} - \frac{1}{2}G_{AB}R = \begin{pmatrix} -\frac{3}{2}(\partial_y^2 a)\eta_{\alpha\beta} & 0 \\ 0 & \frac{3}{2}a^{-2}(\partial_y a)^2 \end{pmatrix} \tag{3.85}$$

Without an additional cosmological constant or matter content, the Einstein field equations demand that the Einstein tensor vanish ($\mathcal{G}_{AB} = 0$). This implies $(\partial_y^2 a) = (\partial_y a) = 0$, which is only achievable by setting a to a constant; however, a constant a just describes the flat 5D metric up to a coordinate rescaling. We desire a more interesting geometry.

By adding a cosmological constant throughout 5D spacetime (a "bulk" cosmological constant), we instead obtain $\mathcal{G}_{AB} - \Lambda G_{AB} = 0$ as our Einstein field equations, wherein the precise value of Λ can be tuned as necessary. Now our constraints read

$$-\frac{3}{2}(\partial_y^2 a) - \Lambda a = 0 \tag{3.86}$$

$$\frac{3}{2}(\partial_y a)^2 + \Lambda a^2 = 0 \tag{3.87}$$

We focus on this second equation first. Immediately, we note a solution cannot exist if $\Lambda > 0$ because $(\partial_y a)^2/a^2$ is necessarily nonnegative. This plus the fact that we already ruled out the $\Lambda = 0$ case as being uninteresting leaves us to consider $\Lambda < 0$, which allows us to solve for $(\partial_y a)$ up a sign: $(\partial_y a) = \pm(\sqrt{-2\Lambda/3})a$, corresponding to $a(y) \propto e^{\pm(\sqrt{-2\Lambda/3})y}$. Define the so-called warping parameter $k \equiv \sqrt{-\Lambda/6}$ for ease of writing, and remove the proportionality so that $a(y) = e^{\pm 2ky}$ via coordinate rescaling.[2] Because this solution also satisfies the first constraint, all may seem well. However, this solution does not respect the orbifold reflection symmetry: neither solution is individually invariant under the replacement $y \to -y$. To fix this, we can patch together separate solutions in the regions $y < 0$ and $y > 0$ to form the continuous & orbifold-even solution $a(y) = e^{\pm 2k|y|}$. Differentiating this new solution yields, keeping in mind the orbifold symmetry and periodic nature of the extra dimension,

$$(\partial_y a)^2 = [\pm 2k\,\mathrm{sign}(y)\,a]^2 = 4k^2\,a^2 \tag{3.88}$$

$$(\partial_y^2 a) = \Big[4k^2 \pm 4k\,(\delta_0 - \delta_{\pi r_c})\Big]\,a \tag{3.89}$$

where we used

$$(\partial_y|y|) = \mathrm{sign}(y) \qquad\qquad (\partial_y^2|y|) = 2\,(\delta_0 - \delta_{\pi r_c}) \tag{3.90}$$

and $\delta_{\overline{y}} \equiv \delta(y - \overline{y})$. Although this orbifold-even solution solves Eq. (3.87), it does not solve Eq. (3.86). In fact, any attempt to modify the action (and therein the Einstein field equations) that treats all of 5D spacetime on equal footing is doomed to fail. We will need to further extend the types of terms we include in the action in order to overcome this difficulty

To better understand why we are running into trouble, let us divide the 5D RS1 spacetime having coordinates (x, y) into a collection of constant y slices, e.g. hypersurfaces consisting of points $(x, \overline{y})$ for some $\overline{y} \in [0, \pi r_c]$. This defines what is called a "foliation" of 5D RS1 spacetime into time-like 4D hypersurfaces, where the hypersurfaces at $\overline{y} = 0$ and $\overline{y} = \pi r_c$ are the RS1 branes. Choose one such hypersurface in this foliation. Because this hypersurface is itself a submanifold of spacetime, we can calculate its curvature. Furthermore, because it exists within a larger spacetime, it has two kinds of curvature: intrinsic (curvature tangent to the hypersurface) and extrinsic (curvature normal to the hypersurface). The extrinsic curvature of a hypersurface is

$$K_{MN} = -\frac{1}{2}\overline{G}_{MP}\,\overline{G}_{NQ}\,\tilde{G}^{PR}\,\tilde{G}^{QS}\,\pounds_n \overline{G}_{RS} \tag{3.91}$$

[2]The metric corresponding to $a(y) = e^{-2ky}$ describes 5D anti-de Sitter space (AdS_5). More specifically, because the RS1 model has branes at $y = 0$ and $y = \pi r_c$, the RS1 model is a finite interval of AdS_5, wherein the brane at $y = \pi r_c$ explicitly breaks the conformal invariance of the infinite AdS_5.

where n^M is a vector field of unit 5-vectors normal to our hypersurface, $\pounds_n$ denotes the Lie derivative along n^M, and $\overline{G}$ is the projection of the metric G onto the hypersurface at $y = \overline{y}$. By choosing $n^M \equiv (0,0,0,0,1)$ as our hypersurface normals, the projected metric equals

$$\overline{G}_{MN}(\overline{y}) = G_{MN}(\overline{y}) + n_M n_N = \begin{pmatrix} a(\overline{y})\,\eta_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix} \tag{3.92}$$

Thus $\overline{G}_{MN}\tilde{G}^{NR} = (\delta^\mu_M \delta^R_\rho)\delta^\rho_\mu$ and the extrinsic curvature simplifies to

$$K_{MN} = -\frac{1}{2}\pounds_n \overline{G}_{MN} \tag{3.93}$$

When acting on a rank-2 covariant tensor such as $\overline{G}_{MN}$, the Lie derivative $\pounds_n$ equals

$$\pounds_n \overline{G}_{MN} = n^A(\partial_A \overline{G}_{MN}) + (\partial_M n^A)\overline{G}_{AN} + (\partial_N n^A)\overline{G}_{AM} \tag{3.94}$$

As parameterized above, n^A does not depend on the coordinates, so

$$\pounds_n \overline{G}_{MN} = n^A(\partial_A \overline{G}_{MN}) = (\partial_y \overline{G}_{MN}) = \begin{pmatrix} (\partial_y a)\eta_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix} \tag{3.95}$$

such that the extrinsic curvature of a constant y hypersurface in a spacetime with metric Eq. (3.76) equals

$$K_{MN}(\overline{y}) = -\frac{1}{2}(\partial_y a)\,\delta^\mu_M \delta^\nu_N\,\eta_{\mu\nu} \tag{3.96}$$

This extrinsic curvature poses a problem when trying to solve the Einstein field equations in the presence of an orbifold-even function like $a(y) = e^{\pm 2k|y|}$. In this case, K_{MN} is nonzero, and thus necessarily implies additional warping in the spacetime geometry not accounted for solely by the standard Einstein-Hilbert Lagrangian nor an additional bulk cosmological constant. In particular, the orbifold symmetry demands that $K_{MN}(0^+) = -K_{MN}(0^-)$ across the orbifold fixed point at $y = 0$ and $K_{MN}(r_c^-) = -K_{MN}((-r_c)^+)$ across the orbifold fixed point at $y = r_c$, which subsequently imply jumps in the extrinsic curvature at the branes, i.e.

$$[K_{MN}]|_{y=\overline{y}} \equiv K^{MN}(\overline{y}^+) - K^{MN}(\overline{y}^-) \tag{3.97}$$

$$= 2K^{MN}(\overline{y}^+) \tag{3.98}$$

$$= -\,(\partial_y a)\big|_{\overline{y}^+\to\overline{y}}\;\delta^\mu_M \delta^\nu_N \eta_{\mu\nu} \tag{3.99}$$

To accomplish a jump in the extrinsic curvature like this, we need a surface source of stress-energy (not unlike using a surface charge density to cause a jump in the electric field in classical E&M). In analogy with our previous (bulk-based) situation, we have two immediate options for trying to achieve this: either embedding matter into the branes, or introducing a surface cosmological constant on each brane. We opt for the latter to keep things purely gravitational, and call each of these new surface cosmological constants a brane tension.

As far as the Einstein field equations are concerned, this means introducing new terms into the action. For terms evaluated on the brane, we use the appropriate brane-projected metric $\overline{G}$, but otherwise the new brane tension terms closely resemble our bulk cosmological constant term: we include them in our existing cosmological constant Lagrangian like so,

$$S_{\text{CC}} = -\frac{4}{\kappa_{5\text{D}}^2} \int d^5x \left[\Lambda\sqrt{G} + \lambda_0 \sqrt{\overline{G}(0)}\, \delta(y) + \lambda_{\pi r_c} \sqrt{\overline{G}(\pi r_c)}\, \delta(y - \pi r_c) \right] \tag{3.100}$$

The constants Λ, λ_0, and $\lambda_{\pi r_c}$ will be determined soon using the Einstein field equations. The variation of the new terms with respect to $\tilde{G}^{MN}$ proceeds similarly to the bulk cosmological constant term so long as we are careful to continue projecting onto each respective brane: using the Lagrangian implied by Eq. (3.100), we find

$$\frac{\delta}{\delta \tilde{G}^{AB}} \left[\mathcal{L}_{\text{CC}}\right] = -\frac{2}{\kappa_{5\text{D}}^2} \left[-\Lambda\sqrt{G}\, G_{AB} - \sum_{\overline{y} \in \{0, \pi r_c\}} \lambda_{\overline{y}} \sqrt{\overline{G}(\overline{y})}\, \overline{G}_{AB}(\overline{y})\, \delta_{\overline{y}} \right] \tag{3.101}$$

where $\delta_{\overline{y}} \equiv \delta(y - \overline{y})$. Therefore, the Einstein fields equations derived from combining Eq. (3.100) and the usual Einstein-Hilbert Lagrangian (in the absence of matter) are

$$\sqrt{G}\left[\mathcal{G}_{AB} - \Lambda G_{AB}\right] - \sum_{\overline{y} \in \{0, \pi r_c\}} \lambda_{\overline{y}} \sqrt{\overline{G}(\overline{y})}\, \overline{G}_{AB}(\overline{y})\, \delta_{\overline{y}} = 0 \tag{3.102}$$

After substituting explicit values into the Einstein field equations Eq. (3.102), including

$$\sqrt{G} = a(y)^2 \qquad\qquad \sqrt{\overline{G}(\overline{y})} = a(\overline{y})^2 \tag{3.103}$$

we obtain

$$-\tfrac{3}{2} a^2 (\partial_y^2 a) - \Lambda\, a^3 - \sum_{\overline{y} \in \{0, \pi r_c\}} \lambda_{\overline{y}}\, a(\overline{y})^3 \delta_{\overline{y}} = 0 \tag{3.104}$$

$$\frac{3}{2}(\partial_y a)^2 + \Lambda a^2 = 0 \tag{3.105}$$

The second equation was solved previously and led us (after a coordinate rescaling) to the orbifold-even function $a(y) = e^{\pm 2k|y|}$ and bulk cosmological constant $\Lambda = -6k^2$. When this solution is substituted into the first equation, all terms lacking Dirac deltas are automatically cancelled, and the residual Dirac deltas only cancel if

$$\mp 6k - \lambda_0 = 0 \qquad\qquad \pm 6k - \lambda_{\pi r_c} = 0 \tag{3.106}$$

Hence, each brane requires a different-signed tension, where the sign of the exponential in $a(y)$ determines which brane gets which sign. It is conventional to choose the sign such that the $y = 0$ brane (sometimes called the hidden or Planck brane) has positive tension and the $y = \pi r_c$ brane (sometimes called the visible or TeV brane) has negative tension. Thus, we choose the lower sign option and find the Einstein field equations are solved by taking

$$a(y) = e^{-2k|y|} \qquad\qquad \Lambda = -k\lambda_0 = k\lambda_{\pi r_c} = -6k^2 \tag{3.107}$$

This completes the construction of the RS1 background metric.

We now summarize the results of the above derivation, but add the label "(bkgd)" while doing so as to emphasize that these results are specific to the RS1 background metric. The background metric 5D RS1 Lagrangian equals

$$\mathcal{L}_{\text{5D}}^{(\text{bkgd})} = \mathcal{L}_{\text{EH}}^{(\text{bkgd})} + \mathcal{L}_{\text{CC}}^{(\text{bkgd})} = -\frac{2}{\kappa_{\text{5D}}^2}\left[\sqrt{G^{(\text{bkgd})}}\, R - 12k^2\sqrt{G^{(\text{bkgd})}} + 6k\sqrt{\overline{G}^{(\text{bkgd})}}\,(\partial_y^2|y|)\right] \tag{3.108}$$

wherein the Einstein-Hilbert and cosmological constant Lagrangians equal

$$\mathcal{L}_{\text{EH}}^{(\text{bkgd})} = -\frac{2}{\kappa_{\text{5D}}^2}\sqrt{G^{(\text{bkgd})}}\, R^{(\text{bkgd})} \tag{3.109}$$

$$\mathcal{L}_{\text{CC}}^{(\text{bkgd})} = \frac{12k}{\kappa_{\text{5D}}^2}\left[6k\sqrt{G^{(\text{bkgd})}} + \sqrt{\overline{G}^{(\text{bkgd})}}(\partial_y^2|y|)\right] \tag{3.110}$$

with corresponding background metric and 4D projection

$$G^{(\text{bkgd})} = \begin{pmatrix} e^{-2k|y|}\eta_{\mu\nu} & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \overline{G}^{(\text{bkgd})} = \begin{pmatrix} e^{-2k|y|}\eta_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix} \tag{3.111}$$

In order to obtain a particle theory of RS1 gravity, we must perturb the background solution summarized in Eqs. (3.108)-(3.111) by field-dependent amounts. This is the topic of the next subsection.

3.3.2 Perturbing The Background Metric

The last subsection constructed the RS1 background metric, which is ultimately described by Eqs. (3.108)-(3.111). The particle theory is then obtained by perturbing this background metric, but we must take care to correctly distinguish physical and unphysical degrees of freedom when doing so. For example, one way to parameterize a generic perturbed metric G relative to the background metric $G^{(\text{bkgd})}$ is

$$G = \begin{pmatrix} e^{-2k\left[|y|+\hat{u}(x,y)\right]}\left(\eta_{\mu\nu} + \kappa_{\text{5D}}\hat{h}_{\mu\nu}(x,y)\right) & \hat{\rho}_\mu(x,y) \\ \hat{\rho}_\nu(x,y) & -\left[1+2\hat{u}(x,y)\right]^2 \end{pmatrix} \tag{3.112}$$

in coordinates $x^M = (x^\mu, y)$, where x^μ are the usual 4D coordinates and $y \in [0, \pi r_c]$ is the extra-dimensional spatial coordinate (which is extended to $y \in [-\pi r_c, +\pi r_c]$ by imposing orbifold invariance). Note that Eq. (3.112) recovers $G^{(\text{bkgd})}$ when $\hat{h} = \hat{\rho} = \hat{u} = 0$. Via coordinate transformations, Eq. (3.112) can always be brought into the form

$$G = \begin{pmatrix} e^{-2k\left[|y|+\hat{u}(x,y)\right]}\left(\eta_{\mu\nu} + \kappa_{\text{5D}}\hat{h}_{\mu\nu}(x,y)\right) & 0 \\ 0 & -\left[1+2\hat{u}(x,y)\right]^2 \end{pmatrix} \tag{3.113}$$

where ρ_μ is made to vanish via orbifold symmetry, and $\hat{u}(x,y)$ equals

$$\hat{u}(x,y) \equiv \frac{\kappa_{\text{5D}}\hat{r}(x)}{2\sqrt{6}} e^{+k(2|y|-\pi r_c)} \tag{3.114}$$

in terms of the y-independent field $\hat{r}(x)$ [27]. The 5D fields $\hat{h}(x,y)$ and $\hat{r}(x)$ contain all dynamical degrees of freedom of the RS1 model [27], and will be the source of our 4D particle content in the next chapter. By demanding that ds^2 be invariant under the orbifold symmetry, $\hat{h}_{\mu\nu}(x,y)$ and $\hat{r}(x)$ are necessarily even functions of y; in other words, these fields are "orbifold even." Furthermore, because G_{MN} is symmetric in its indices, $\hat{h}_{\mu\nu}(x,y)$ is symmetric as well.

For convenience, we will often parameterize the perturbed metric G (and its projection onto a constant y hypersurface, $\overline{G}$) as

$$G_{MN} = \begin{pmatrix} w(x,y)\, g_{\mu\nu} & 0 \\ 0 & -v(x,y)^2 \end{pmatrix} \qquad \overline{G}_{MN} = \begin{pmatrix} w(x,y)\, g_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix} \tag{3.115}$$

where

$$g_{\mu\nu}(x,y) \equiv \eta_{\mu\nu} + \kappa_{5\text{D}} h_{\mu\nu}(x,y) \tag{3.116}$$
$$w(x,y) \equiv \varepsilon^{-2} e^{-2\hat{u}(x)} \tag{3.117}$$
$$v(x,y) \equiv 1 + 2\hat{u}(x) \tag{3.118}$$

and $\varepsilon \equiv e^{-k|y|}$. Replacing $G^{(\text{bkgd})}$ with G (and $\overline{G}^{(\text{bkgd})}$ with $\overline{G}$) in Eqs. (3.108)-(3.111) yields the 5D RS1 theory:

$$\mathcal{L}_{5\text{D}} = \mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{CC}} \tag{3.119}$$

where

$$\mathcal{L}_{\text{EH}} \equiv -\frac{2}{\kappa^2}\sqrt{G}\, R \cong -\frac{2}{\kappa_{5\text{D}}^2}\sqrt{G}\tilde{G}^{MN}\left[\Gamma^{Q}_{MP}\Gamma^{P}_{NQ} - \Gamma^{Q}_{MN}\Gamma^{P}_{PQ}\right] \tag{3.120}$$
$$\mathcal{L}_{\text{CC}} = -\frac{2}{\kappa^2}\left[-12k^2\sqrt{G} + 6k\sqrt{\overline{G}}\,(\partial_y^2|y|)\right] \tag{3.121}$$

The alternate form of $\mathcal{L}_{\text{EH}}$ included on the RHS of Eq. (3.120) was derived in Subsection 3.2.6.

In this parameterization, the invariant interval equals

$$ds^2 = (G_{MN})\, dx^M\, dx^N = (w\, g_{\mu\nu})\, dx^\mu\, dx^\nu - (v^2)\, dy^2 \tag{3.122}$$

where $\tilde{g}^{\mu\nu} g_{\nu\rho} = \mathbb{1}^{\mu}_{\rho}$. Furthermore, the inverse metric $\tilde{G}^{MN}$ equals

$$[\tilde{G}^{MN}] = \begin{pmatrix} w(x,y)^{-1}\, \tilde{g}^{\mu\nu} & 0 \\ 0 & -v(x,y)^{-2} \end{pmatrix} \tag{3.123}$$

where $\tilde{g}^{\mu\nu}$ is the inverse of $g_{\mu\nu} = \eta_{\mu\nu} + \kappa_{5\text{D}}\hat{h}_{\mu\nu}$, and the invariant volume element nicely decomposes into a four-dimensional and extra-dimensional weights:

$$\sqrt{\det G}\, d^4x\, dy = \left[w^2\sqrt{-\det g}\, d^4x\right]\cdot(v\, dy) \tag{3.124}$$

For use in the next subsection, note that $(\partial_y u) = +2k(\partial_y|y|)u$, such that

$$(\partial_y w) = -2k(\partial_y|y|)\, w + 2(\partial_y u)\, w \tag{3.125}$$
$$= -2k(\partial_y|y|)\,(1+2u)\, w \tag{3.126}$$
$$= -2k(\partial_y|y|)\, v\, w \tag{3.127}$$

and, thus,

$$\partial_y G_{\mu\nu} = \partial_y(w\, g_{\mu\nu}) \tag{3.128}$$
$$= (\partial_y w) g_{\mu\nu} + w(\partial_y g_{\mu\nu}) \tag{3.129}$$
$$= 2k(\partial_y|y|)\, v\, w + w(\partial_y g_{\mu\nu}) \tag{3.130}$$

In order to eventually obtain the 4D effective RS1 model, its particle content, and its interactions (which are necessary to analyze the processes in which we are interested), we must weak field expand (WFE) the 5D RS1 Lagrangian. That is, we must series expand the 5D RS1 Lagrangian in powers of the 5D fields $\hat{h}_{\mu\nu}$ and $\hat{r}$. In principle, we could begin the weak field expansion now, but it is worthwhile to first modify $\mathcal{L}_{\text{5D}}$ by the addition of a total derivative $\Delta\mathcal{L}$ which will eliminate any terms proportional to $(\partial_y|y|)$ and $(\partial_y^2|y|)$ from the Lagrangian. This is achieved in the next subsection.

3.3.3 Eliminating "Cosmological Constant"-Like Terms

The cosmological constant Lagrangian Eq. (3.121) contains terms that potentially complicate our analysis. For example, the terms proportional to $(\partial_y^2|y|)$ introduce Dirac deltas. When going from the 5D theory to the 4D effective theory in the next chapter, we will integrate the Lagrangian over the extra dimension, such that the presence of Dirac deltas will replace what would otherwise become coupling integrals with evaluations of extra-dimensional wavefunctions at the branes. Thankfully, such terms in the cosmological constant Lagrangian combine with similar terms in the Einstein-Hilbert Lagrangian Eq. (3.120) to form physically-irrelevant total derivatives, and in doing so all terms proportional to $(\partial_y|y|)$ or $(\partial_y^2|y|)$ are eliminated. The present subsection will explicitly demonstrate the elimination of these terms to all orders in the 5D fields as well as introducing a new term $\Delta\mathcal{L}$ to the RS1 Lagrangian which automates this elimination.

The terms in $\mathcal{L}_{\text{EH}}$ which cancel $\mathcal{L}_{\text{CC}}$ arise when an extra-dimensional derivative ∂_y acts on a y-dependent multiplicative factor such as ε or $(\partial_y|y|)$ instead of the 5D field $\hat{h}_{\mu\nu}$ (recall that $\hat{r}$ is y-independent by construction). Hence, for the purposes of this subsection, we seek to isolate all such terms in $\mathcal{L}_{\text{EH}}$. To begin, we recalculate the Christoffel symbols (originally calculated in Eqs. (3.77)-(3.78) for the RS1 background solution) for the perturbed theory: recall

$$\Gamma^P_{MN} \equiv \frac{1}{2}\tilde{G}^{PQ}(\partial_M G_{NQ} + \partial_N G_{MQ} - \partial_Q G_{MN}) \tag{3.131}$$

such that, using the fact that G_{MN} and its inverse $\tilde{G}^{MN}$ are block-diagonal,

$$\Gamma^{5}_{\mu\nu} = -\frac{1}{2}\tilde{G}^{55}(\partial_5 G_{\mu\nu}) \tag{3.132}$$

$$\Gamma^{\rho}_{5\nu} = +\frac{1}{2}\tilde{G}^{\rho\sigma}(\partial_5 G_{\nu\sigma}) \qquad \Longrightarrow \qquad \Gamma^{\rho}_{5\rho} = +\frac{1}{2}[\![\tilde{G}G']\!] \tag{3.133}$$

$$\Gamma^{5}_{5\nu} = +\frac{1}{2}\tilde{G}^{55}(\partial_\nu G_{55}) \tag{3.134}$$

$$\Gamma^{\rho}_{55} = -\frac{1}{2}\tilde{G}^{\rho\sigma}(\partial_\sigma G_{55}) \tag{3.135}$$

$$\Gamma^{5}_{55} = +\frac{1}{2}\tilde{G}^{55}(\partial_5 G_{55}) \tag{3.136}$$

where $\partial_5 \equiv \partial_y$. Because $\tilde{G}^{MN}$ is block-diagonal, the index summations on the RHS of Eq. (3.120) only yield nonzero contributions when $(M,N) = (\mu,\nu)$ and $(M,N) = (5,5)$. Consider when $(M,N) = (\mu,\nu)$. The first product of Christoffel symbols in the $(M,N) = (\mu,\nu)$ case equals

$$\Gamma^{Q}_{\mu P}\Gamma^{P}_{\nu Q} = \Gamma^{\sigma}_{\mu\rho}\Gamma^{\rho}_{\nu\sigma} + \Gamma^{5}_{\mu\rho}\Gamma^{\rho}_{\nu 5} + \Gamma^{\sigma}_{\mu 5}\Gamma^{5}_{\nu\sigma} + \Gamma^{5}_{\mu 5}\Gamma^{5}_{\nu 5} \tag{3.137}$$

of which the second and third terms contain y-derivatives. Their contributions are identical and yield, when combined,

$$\Gamma^{Q}_{\mu P}\Gamma^{P}_{\nu Q} \overset{\partial_y \text{ not on}}{\underset{\text{a field}}{\supset}} -\frac{1}{2}\tilde{G}^{55}[\![G'\tilde{G}G']\!]_{\mu\nu} \tag{3.138}$$

The second product of Christoffel symbols in the $(M,N) = (\mu,\nu)$ case equals

$$\Gamma^{Q}_{\mu\nu}\Gamma^{P}_{PQ} = \Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\rho\sigma} + \Gamma^{5}_{\mu\nu}\Gamma^{\rho}_{\rho 5} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{5}_{5\sigma} + \Gamma^{5}_{\mu\nu}\Gamma^{5}_{55} \tag{3.139}$$

of which the second and fourth terms contain y-derivatives, such that

$$\Gamma^{Q}_{\mu\nu}\Gamma^{P}_{PQ} \overset{\partial_y \text{ not on}}{\underset{\text{a field}}{\supset}} -\frac{1}{4}\tilde{G}^{55}[\![\tilde{G}G']\!](\partial_5 G_{\mu\nu}) - \frac{1}{4}\tilde{G}^{55}\tilde{G}^{55}(\partial_5 G_{55})(\partial_5 G_{\mu\nu}) \tag{3.140}$$

Hence, when contracted with $\tilde{G}^{\mu\nu}$, the net contributions coming from the $(M,N) = (\mu,\nu)$ case equal

$$\begin{aligned}\tilde{G}^{\mu\nu}\left[\Gamma^{Q}_{\mu P}\Gamma^{P}_{\nu Q} - \Gamma^{Q}_{\mu\nu}\Gamma^{P}_{PQ}\right] \overset{\partial_y \text{ not on}}{\underset{\text{a field}}{\supset}} & -\frac{1}{2}\tilde{G}^{55}[\![\tilde{G}G'\tilde{G}G']\!] + \frac{1}{4}\tilde{G}^{55}[\![\tilde{G}G']\!]^2 \\ & + \frac{1}{4}\tilde{G}^{55}\tilde{G}^{55}(\partial_5 G_{55})[\![\tilde{G}G']\!]\end{aligned} \tag{3.141}$$

Meanwhile, the equivalent expression in the $(5,5)$ case equals, thanks to cancellations,

$$\tilde{G}^{55}\left[\Gamma^{Q}_{5P}\Gamma^{P}_{5Q} - \Gamma^{Q}_{55}\Gamma^{P}_{PQ}\right] = \tilde{G}^{55}\left[\Gamma^{\sigma}_{5\rho}\Gamma^{\rho}_{5\sigma} + \Gamma^{5}_{5\rho}\Gamma^{\rho}_{55} - \Gamma^{\sigma}_{55}\Gamma^{\rho}_{\rho\sigma} - \Gamma^{5}_{55}\Gamma^{\rho}_{\rho 5}\right] \tag{3.142}$$

of which the first and third terms contain y-derivatives, contributing overall

$$\tilde{G}^{55}\left[\Gamma^{Q}_{5P}\Gamma^{P}_{5Q} - \Gamma^{Q}_{55}\Gamma^{P}_{PQ}\right] \overset{\partial_y \text{ not on}}{\underset{\text{a field}}{\supset}} +\frac{1}{4}\tilde{G}^{55}[\![\tilde{G}G'\tilde{G}G']\!] - \frac{1}{4}\tilde{G}^{55}\tilde{G}^{55}(\partial_5 G_{55})[\![\tilde{G}G']\!] \tag{3.143}$$

Combining Eqs. (3.141) and (3.143) yields, at the level of the Einstein-Hilbert Lagrangian,

$$\mathcal{L}_{\text{EH}} \overset{\partial_y \text{ not on}}{\underset{\text{a field}}{\supset}} -\frac{2}{\kappa^2_{5\text{D}}}\sqrt{G}\,\tilde{G}^{55}\left[\frac{1}{4}[\![\tilde{G}G']\!]^2 - \frac{1}{4}[\![\tilde{G}G'\tilde{G}G']\!]\right] \tag{3.144}$$

However, this expression contains more than just the terms we desire: some of the y-derivatives in this expression will end up acting on fields and, thus, not help eliminate $\mathcal{L}_{\text{CC}}$. To refine this expression further, we utilize the explicit form of G, Eq. (3.115). For example, with this parameterization the prefactor $\sqrt{G}\,\tilde{G}^{55}$ becomes $(v\,w^2\sqrt{-g})(-1/v^2) = -(w^2/v)\sqrt{-g}$. This decomposition also allows $[\![\tilde{G}G']\!]$ to be rewritten as

$$[\![\tilde{G}G']\!] = 4(\partial_y \ln w) + [\![\tilde{g}g']\!] \tag{3.145}$$

where we utilized Eq. (3.129) and the fact that $[\![\tilde{G}G]\!] = [\![\eta]\!] = 4$. Squaring this, we then obtain

$$[\![\tilde{G}G']\!]^2 = 16(\partial_y \ln w)^2 + 8(\partial_y \ln w)[\![\tilde{g}g']\!] + [\![\tilde{g}g']\!]^2 \tag{3.146}$$

The final term in Eq. (3.146) only contains y-derivatives acting on fields and thus can be ignored from here on. Similarly, the second term in Eq. (3.144) is proportional to

$$[\![\tilde{G}G'\tilde{G}G']\!] = 4(\partial_y \ln w)^2 + 2(\partial_y \ln w)[\![\tilde{g}g']\!] + [\![\tilde{g}g'\tilde{g}g']\!] \tag{3.147}$$

wherein the first two terms involve $(\partial_y w) \propto (\partial_y|y|)$ and the final term can be ignored. By keeping these distinctions in mind, the only terms in $\mathcal{L}_{\text{EH}}$ where y-derivatives do not act on fields are

$$\mathcal{L}_{\text{EH}} \overset{\partial_y \text{ not on}}{\underset{\text{a field}}{\supset}} -\frac{2}{\kappa^2_{5\text{D}}}\left(-\frac{w^2}{v}\sqrt{-g}\right)\left[3(\partial_y \ln w)^2 - \frac{3}{2}(\partial_y \ln w)[\![\tilde{g}g']\!]\right] \tag{3.148}$$

But $(\partial_y \ln w) = (\partial_y w)/w = 2k(\partial_y|y|)\,v$ via Eq. (3.127), such that

$$\mathcal{L}_{\text{EH}} \overset{\partial_y \text{ not on}}{\underset{\text{a field}}{\supset}} -\frac{2}{\kappa^2_{5\text{D}}}w^2\sqrt{-g}\left[-12k^2 v + 3k(\partial_y|y|)[\![\tilde{g}g']\!]\right] \tag{3.149}$$

This completes our manipulations of the Einstein-Hilbert Lagrangian. We can apply a similar decomposition to $\mathcal{L}_{\text{CC}}$ in Eq. (3.121):

$$\mathcal{L}_{\text{CC}} = -\frac{2}{\kappa^2_{5\text{D}}}w^2\sqrt{-g}\left[-12k^2 v + 6k(\partial_y^2|y|)\right] \tag{3.150}$$

where $\sqrt{\overline{G}} = w^2\sqrt{-g}$ because $\overline{G}$ only includes the 4-by-4 part of the metric G. Combining Eq. (3.150) in its entirety with the terms we isolated from $\mathcal{L}_{\text{EH}}$ in Eq. (3.149) yields, in total,

$$\mathcal{L}_{\text{5D}} = \mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{CC}} \overset{\partial_y \text{ not on}}{\underset{\text{a field}}{\supset}} -\frac{6k}{\kappa_{\text{5D}}^2} w^2\sqrt{-g}\left[-8kv + (\partial_y|y|)[\![\tilde{g}g']\!] + 2(\partial_y^2|y|)\right] \tag{3.151}$$

Thankfully, this collection of terms actually forms the total derivative $\partial_y[w^2\sqrt{-g}\,(\partial_y|y|)]$ up to multiplicative constants:

$$\begin{aligned}\partial_y\left[w^2\sqrt{-g}\,(\partial_y|y|)\right] &= \sqrt{-g}\left[2w(\partial_y w)(\partial_y|y|) + \frac{1}{2}w^2[\![\tilde{g}g']\!](\partial_y|y|) + w^2\,(\partial_y^2|y|)\right] &(3.152)\\ &= \frac{1}{2}w^2\sqrt{-g}\left[4(\partial_y \ln w)(\partial_y|y|) + (\partial_y|y|)[\![\tilde{g}g']\!] + 2(\partial_y^2|y|)\right] &(3.153)\\ &= \frac{1}{2}w^2\sqrt{-g}\left[-8kv + (\partial_y|y|)[\![\tilde{g}g']\!] + 2(\partial_y^2|y|)\right] &(3.154)\end{aligned}$$

Therefore, as desired all terms in $\mathcal{L}_{\text{5D}}$ that resemble contributions from the cosmological constant Lagrangian combine to form a total derivative,

$$\mathcal{L}_{\text{5D}} = \mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{CC}} \overset{\partial_y \text{ not on}}{\underset{\text{a field}}{\supset}} -\frac{12k}{\kappa_{\text{5D}}^2}\partial_y\left[w^2\sqrt{-g}\,(\partial_y|y|)\right] \cong 0 \tag{3.155}$$

and only terms where derivatives are applied to fields contribute to the physics.[3] Furthermore, because of the structure of $\mathcal{L}_{\text{EH}}$ in Eq. (3.120), there are two derivatives in every term and those derivatives never act on the same field instance. This fact is useful in the next chapter, when we analyze the coupling structures present in the 4D effective RS1 theory.

To avoid performing the integration by parts implied by Eq. (3.155) in the future, we can manually subtract the total derivative we eliminated from the 5D Lagrangian and use, in practice,

$$\mathcal{L}_{\text{5D}} = \mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{CC}} + \Delta\mathcal{L} \tag{3.156}$$

where[4]

$$\Delta\mathcal{L} \equiv \frac{12k}{\kappa_{\text{5D}}^2}\partial_y\left[w^2\sqrt{-g}\,(\partial_y|y|)\right] \tag{3.157}$$

With this result, we now weak field expand the 5D RS1 Lagrangian.

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[3] That the total derivative does not contribute a nonzero surface term to the action is guaranteed by the discrete translation invariance of the integral, such that whatever contribution we obtain from a boundary term at $y = +\pi r_c$ is exactly cancelled by an identical term at $y = -\pi r_c$.

[4] This $\Delta\mathcal{L}$ is different than the $\Delta\mathcal{L}$ used in [18] because the present dissertation uses the alternate form for $\mathcal{L}_{\text{EH}}$ derived in Subsection 3.2.6 as opposed to its more traditional form.

3.4 5D Weak Field Expanded RS1 Lagrangian[5]

3.4.1 Notation

This section details the weak field expansion of the RS1 model Lagrangian, Eq. (3.156), including explicit expressions for all terms in the Lagrangian having four or fewer instances of the 5D fields $\hat{h}_{\mu\nu}(x,y)$ and $\hat{r}(x)$. The matter-free RS1 model Lagrangian $\mathcal{L}_{\text{5D}}^{(\text{RS})}$ is defined by Eq. (3.156) and series expanded in terms of the 5D fields via Eqs. (3.114)-(3.118). The resulting terms are then sorted according to 5D field content:

$$\begin{aligned}\mathcal{L}_{\text{5D}}^{(\text{RS})} = \mathcal{L}_{hh}^{(\text{RS})} + \mathcal{L}_{rr}^{(\text{RS})} + \mathcal{L}_{hhh}^{(\text{RS})} + \cdots + \mathcal{L}_{rrr}^{(\text{RS})} \\ + \mathcal{L}_{hhhh}^{(\text{RS})} + \cdots + \mathcal{L}_{rrrr}^{(\text{RS})} + \cdots \ .\end{aligned} \tag{3.158}$$

Primes on $\hat{h}_{\mu\nu}$ indicate derivatives with respect to y. A product of $\hat{h}_{\mu\nu}$ fields with Lorentz indices contracted to form a chain is indicated via twice-squared bracket notation, e.g.

$$[\![\hat{h}']\!] \equiv (\partial_y \hat{h}^{\alpha}_{\alpha}) \qquad [\![\hat{h}\hat{h}]\!]_{\alpha\beta} = \hat{h}_{\alpha\gamma}\hat{h}^{\gamma}_{\beta} \qquad [\![\hat{h}\hat{h}\hat{h}]\!] = \hat{h}^{\alpha}_{\beta}\hat{h}^{\beta}_{\gamma}\hat{h}^{\gamma}_{\alpha} \tag{3.159}$$

We also utilize the following abbreviations

$$\hat{h} \equiv \hat{h}^{\alpha}_{\alpha} \qquad (\partial_\alpha \hat{h}) \equiv (\partial_\alpha \hat{h}^{\beta}_{\beta}) \qquad (\partial\hat{h})_\alpha \equiv (\partial^\beta \hat{h}_{\beta\alpha}) \tag{3.160}$$

when writing the Lagrangians.

The 4D metric g exactly satisfies

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \kappa \hat{h}_{\alpha\beta} \ . \tag{3.161}$$

From this, the 4D inverse metric $\tilde{g}$ may be solved for order-by-order by imposing its defining condition, $g_{\alpha\beta}\tilde{g}^{\beta\gamma} = \eta^{\gamma}_{\alpha}$, which implies

$$\tilde{g}^{\alpha\beta} = \eta^{\alpha\beta} + \sum_{n=1}^{+\infty} (-\kappa)^n [\![\hat{h}^n]\!]^{\alpha\beta} \ . \tag{3.162}$$

Meanwhile, the 4D determinant equals

$$\sqrt{-\det \tilde{g}} = \prod_{n=1}^{+\infty} \exp\left[\frac{(-1)^{n-1}}{2n}\kappa^n \, [\![\hat{h}^n]\!]\right] \ . \tag{3.163}$$

which yields, for example,

$$\begin{aligned}\sqrt{-\det \tilde{g}} = 1 + \frac{\kappa}{2}\,\hat{h} + \frac{\kappa^2}{8}\left(\hat{h}^2 - 2[\![\hat{h}\hat{h}]\!]\right) + \frac{\kappa^3}{48}\left(\hat{h}^3 - 6\hat{h}[\![\hat{h}\hat{h}]\!] + 8[\![\hat{h}\hat{h}\hat{h}]\!]\right) \\ + \frac{1}{384}\left(\hat{h}^4 - 12\hat{h}^2[\![\hat{h}\hat{h}]\!] + 12[\![\hat{h}\hat{h}]\!]^2 + 32\hat{h}[\![\hat{h}\hat{h}\hat{h}]\!] - 48[\![\hat{h}\hat{h}\hat{h}\hat{h}]\!]\right) + \mathcal{O}(\kappa^5) \ .\end{aligned}$$

[5]This section was originally published as Appendix A of [18]. The content has been updated to reflect the new form of the Einstein-Hilbert Lagrangian, and material has been added to connect this section to the rest of this dissertation.

Finally, separating the interactions that involve y derivatives from the interactions that do not, which we call B-type and A-type interactions respectively, we define $\overline{\mathcal{L}}_A$ and $\overline{\mathcal{L}}_B$ according to the following decomposition:

$$\mathcal{L}^{(\text{RS})}_{hH_rR} = \kappa^{H+R-2} \left[e^{-\pi k r_c}\varepsilon^{+2}\right]^R \left[\varepsilon^{-2}\overline{\mathcal{L}}_{A:hH_rR} + \varepsilon^{-4}\overline{\mathcal{L}}_{B:hH_rR}\right] , \tag{3.164}$$

where $\varepsilon \equiv e^{-kr_c|\varphi|}$. There is then the question: to what order in what fields should the 5D RS1 Lagrangian be expanded? For the processes relevant to this dissertation (tree-level 2-to-2 scattering), we require the quartic $\hat{h}\hat{h}\hat{h}\hat{h}$ interaction, which occurs at $\mathcal{O}(\kappa^2)$. Because we have already calculated them anyway, we provide all terms in the weak field expansion of the RS1 Lagrangian that occur at $\mathcal{O}(\kappa^2)$ and lower. These interaction Lagrangians in principle enable the calculation of all 2-to-2 tree-level scattering matrix elements in the matter-free RS1 model.

The next several subsections summarize these interaction Lagrangians, which are the principle results of this chapter. Afterwards, appendices detail variational derivatives and weak field expansion formulas that wee used in the process of getting these results.

3.4.2 Quadratic-Level Results

$$\overline{\mathcal{L}}_{A:hh} = (\partial\hat{h})_\mu(\partial^\mu\hat{h}) - (\partial\hat{h})^2_\mu + \frac{1}{2}(\partial_\mu\hat{h}_{\nu\rho})^2 - \frac{1}{2}(\partial_\mu\hat{h})^2 \tag{3.165}$$

$$\overline{\mathcal{L}}_{B:hh} = \frac{1}{2}[\![\hat{h}']\!]^2 - \frac{1}{2}[\![\hat{h}'\hat{h}']\!] \tag{3.166}$$

$$\overline{\mathcal{L}}_{A:rr} = \frac{1}{2}(\partial_\mu\hat{r})^2 \tag{3.167}$$

$$\overline{\mathcal{L}}_{B:rr} = 0 \tag{3.168}$$

3.4.3 Cubic-Level Results

$$\begin{aligned}\overline{\mathcal{L}}_{A:hhh} = &\frac{1}{2}\hat{h}(\partial\hat{h})_\mu(\partial^\mu\hat{h}) - \hat{h}_{\mu\nu}(\partial\hat{h})^\mu(\partial^\nu\hat{h}) - \frac{1}{4}\hat{h}(\partial_\mu\hat{h})^2 - \hat{h}_{\nu\rho}(\partial\hat{h})_\mu(\partial^\mu\hat{h}^{\nu\rho}) \\ &+ \hat{h}_{\nu\rho}(\partial_\mu\hat{h})(\partial^\mu\hat{h}^{\nu\rho}) + \frac{1}{4}\hat{h}(\partial_\mu\hat{h}_{\nu\rho})^2 - \hat{h}^\rho_\sigma(\partial_\mu\hat{h}_{\nu\rho})(\partial^\mu\hat{h}^{\nu\sigma}) \\ &+ \frac{1}{2}\hat{h}_{\mu\nu}(\partial^\mu\hat{h})(\partial^\nu\hat{h}) - \hat{h}_{\mu\rho}(\partial^\mu\hat{h}^{\nu\rho})(\partial_\nu\hat{h}) - \frac{1}{2}\hat{h}(\partial_\mu\hat{h}_{\nu\rho})(\partial^\nu\hat{h}^{\mu\rho}) \\ &+ \hat{h}^\sigma_\rho(\partial_\mu\hat{h}_{\nu\sigma})(\partial^\nu\hat{h}^{\mu\rho}) + 2\hat{h}^\sigma_\rho(\partial_\mu\hat{h}_{\nu\sigma})(\partial^\rho\hat{h}^{\mu\nu}) - \frac{1}{2}\hat{h}^\sigma_\rho(\partial_\sigma\hat{h}_{\mu\nu})(\partial^\rho\hat{h}^{\mu\nu})\end{aligned} \tag{3.169}$$

$$\overline{\mathcal{L}}_{B:hhh} = \frac{1}{4}\hat{h}[\![\hat{h}']\!]^2 - [\![\hat{h}']\!][\![\hat{h}\hat{h}']\!] - \frac{1}{4}\hat{h}[\![\hat{h}'\hat{h}']\!] + [\![\hat{h}\hat{h}'\hat{h}']\!] \tag{3.170}$$

$$\overline{\mathcal{L}}_{A:hhr} = 0 \tag{3.171}$$

$$\overline{\mathcal{L}}_{B:hhr} = \frac{1}{2}\sqrt{\frac{3}{2}}\left[[\![\hat{h}'\hat{h}']\!] - [\![\hat{h}']\!]^2\right]\hat{r} \tag{3.172}$$

$$\overline{\mathcal{L}}_{A:hrr} = -\frac{1}{3}(\partial\hat{h})_\mu\hat{r}(\partial^\mu\hat{r}) + \frac{1}{3}(\partial_\mu\hat{h})\hat{r}(\partial^\mu\hat{r}) + \frac{1}{4}\hat{h}(\partial_\mu\hat{r})^2 - \frac{1}{2}\hat{h}_{\mu\nu}(\partial^\mu\hat{r})(\partial^\nu\hat{r}) \tag{3.173}$$

$$\overline{\mathcal{L}}_{B:hrr} = 0 \tag{3.174}$$

$$\overline{\mathcal{L}}_{A:rrr} = -\frac{1}{\sqrt{6}}\hat{r}(\partial_\mu\hat{r})^2 \tag{3.175}$$

$$\overline{\mathcal{L}}_{B:rrr} = 0 \tag{3.176}$$

3.4.4 Quartic-Level Results

$$
\begin{aligned}
\overline{\mathcal{L}}_{A:hhhh} = {} & \frac{1}{8}\hat{h}^2(\partial\hat{h})_\mu(\partial^\mu\hat{h}) - \frac{1}{4}\llbracket\hat{h}\hat{h}\rrbracket(\partial\hat{h})_\mu(\partial^\mu\hat{h}) - \frac{1}{2}\hat{h}\hat{h}_{\mu\nu}(\partial\hat{h})^\mu(\partial^\nu\hat{h}) + \llbracket\hat{h}\hat{h}\rrbracket_{\mu\nu}(\partial\hat{h})^\mu(\partial^\nu\hat{h}) \\
& - \frac{1}{16}\hat{h}^2(\partial_\mu\hat{h})^2 + \frac{1}{8}\llbracket\hat{h}\hat{h}\rrbracket(\partial_\mu\hat{h})^2 - \frac{1}{2}\hat{h}\hat{h}_{\mu\nu}(\partial\hat{h})_\rho(\partial^\rho\hat{h}^{\mu\nu}) + \llbracket\hat{h}\hat{h}\rrbracket_{\mu\nu}(\partial\hat{h})_\rho(\partial^\rho\hat{h}^{\mu\nu}) \\
& + \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial\hat{h})^\mu(\partial^\nu\hat{h}^{\rho\sigma}) + \frac{1}{2}\hat{h}\hat{h}_{\mu\nu}(\partial_\rho\hat{h})(\partial^\rho\hat{h}^{\mu\nu}) - \llbracket\hat{h}\hat{h}\rrbracket_{\mu\nu}(\partial_\rho\hat{h})(\partial^\rho\hat{h}^{\mu\nu}) \\
& + \frac{1}{16}\hat{h}^2(\partial_\rho\hat{h}_{\mu\nu})^2 - \frac{1}{8}\llbracket\hat{h}\hat{h}\rrbracket(\partial_\rho\hat{h}_{\mu\nu})^2 - \frac{1}{2}\hat{h}\hat{h}^\sigma_\rho(\partial_\mu\hat{h}_{\nu\sigma})(\partial^\mu\hat{h}^{\nu\rho}) \\
& + \llbracket\hat{h}\hat{h}\rrbracket^\sigma_\rho(\partial_\mu\hat{h}_{\nu\sigma})(\partial^\mu\hat{h}^{\nu\rho}) - \frac{1}{2}\hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial_\tau\hat{h}^{\mu\nu})(\partial^\tau\hat{h}^{\rho\sigma}) + \frac{1}{2}\hat{h}_{\mu\sigma}\hat{h}_{\rho\nu}(\partial_\tau\hat{h}^{\mu\nu})(\partial^\tau\hat{h}^{\rho\sigma}) \\
& + \frac{1}{4}\hat{h}\hat{h}_{\mu\nu}(\partial^\mu\hat{h})(\partial^\nu\hat{h}) - \frac{1}{2}\llbracket\hat{h}\hat{h}\rrbracket_{\mu\nu}(\partial^\mu\hat{h})(\partial^\nu\hat{h}) - \frac{1}{2}\hat{h}\hat{h}_{\mu\nu}(\partial_\rho\hat{h})(\partial^\mu\hat{h}^{\nu\rho}) \\
& + \llbracket\hat{h}\hat{h}\rrbracket_{\mu\nu}(\partial_\rho\hat{h})(\partial^\mu\hat{h}^{\nu\rho}) + \hat{h}_{\mu\rho}\hat{h}_{\nu\sigma}(\partial^\mu\hat{h})(\partial^\nu\hat{h}^{\rho\sigma}) - \hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial^\mu\hat{h})(\partial^\nu\hat{h}^{\rho\sigma}) \\
& - \frac{1}{8}\hat{h}^2(\partial_\mu\hat{h}_{\nu\rho})(\partial^\nu\hat{h}^{\mu\rho}) + \frac{1}{4}\llbracket\hat{h}\hat{h}\rrbracket(\partial_\mu\hat{h}_{\nu\rho})(\partial^\nu\hat{h}^{\mu\rho}) + \frac{1}{2}\hat{h}\hat{h}^\sigma_\rho(\partial_\mu\hat{h}_{\nu\sigma})(\partial^\nu\hat{h}^{\mu\rho}) \\
& - \llbracket\hat{h}\hat{h}\rrbracket^\sigma_\rho(\partial_\mu\hat{h}_{\nu\sigma})(\partial^\nu\hat{h}^{\mu\rho}) + \hat{h}\hat{h}^\sigma_\rho(\partial_\mu\hat{h}_{\nu\sigma})(\partial^\rho\hat{h}^{\mu\nu}) - 2\llbracket\hat{h}\hat{h}\rrbracket^\sigma_\rho(\partial_\mu\hat{h}_{\nu\sigma})(\partial^\rho\hat{h}^{\mu\nu}) \\
& - 2\hat{h}_{\mu\nu}\hat{h}_{\rho\sigma}(\partial_\tau\hat{h}^{\nu\rho})(\partial^\sigma\hat{h}^{\tau\mu}) + \hat{h}_{\mu\sigma}\hat{h}_{\nu\rho}(\partial_\tau\hat{h}^{\nu\rho})(\partial^\sigma\hat{h}^{\tau\mu}) - \frac{1}{4}\hat{h}\hat{h}^\sigma_\rho(\partial_\sigma\hat{h}_{\mu\nu})(\partial^\rho\hat{h}^{\mu\nu}) \\
& + \frac{1}{2}\llbracket\hat{h}\hat{h}\rrbracket^\sigma_\rho(\partial_\sigma\hat{h}_{\mu\nu})(\partial^\rho\hat{h}^{\mu\nu}) - \hat{h}^{\mu\nu}\hat{h}_{\rho\sigma}(\partial_\mu\hat{h}^{\sigma\tau})(\partial^\rho\hat{h}_{\nu\tau}) + \hat{h}^\mu_\rho\hat{h}^\nu_\sigma(\partial_\mu\hat{h}^{\sigma\tau})(\partial^\rho\hat{h}_{\nu\tau})
\end{aligned}
\tag{3.177}
$$

$$
\begin{aligned}
\overline{\mathcal{L}}_{B:hhhh} = {} & \frac{1}{16}\hat{h}^2\llbracket\hat{h}'\rrbracket^2 - \frac{1}{8}\llbracket\hat{h}\hat{h}\rrbracket\llbracket\hat{h}'\rrbracket^2 - \frac{1}{2}\hat{h}\llbracket\hat{h}'\rrbracket\llbracket\hat{h}\hat{h}'\rrbracket + \llbracket\hat{h}'\rrbracket\llbracket\hat{h}\hat{h}\hat{h}'\rrbracket - \frac{1}{16}\hat{h}^2\llbracket\hat{h}'\hat{h}'\rrbracket \\
& + \frac{1}{8}\llbracket\hat{h}\hat{h}\rrbracket\llbracket\hat{h}'\hat{h}'\rrbracket + \frac{1}{2}\hat{h}\llbracket\hat{h}\hat{h}'\hat{h}'\rrbracket - \llbracket\hat{h}\hat{h}\hat{h}'\hat{h}'\rrbracket + \frac{1}{2}\llbracket\hat{h}\hat{h}'\rrbracket^2 - \frac{1}{2}\llbracket\hat{h}\hat{h}'\hat{h}\hat{h}'\rrbracket
\end{aligned}
\tag{3.178}
$$

$$
\overline{\mathcal{L}}_{A:hhhr} = 0 \tag{3.179}
$$

$$
\overline{\mathcal{L}}_{B:hhhr} = \frac{1}{4}\sqrt{\frac{3}{2}}\Big[-\hat{h}\llbracket\hat{h}'\rrbracket^2 + 4\llbracket\hat{h}'\rrbracket\llbracket\hat{h}\hat{h}'\rrbracket + \hat{h}\llbracket\hat{h}'\hat{h}'\rrbracket - 4\llbracket\hat{h}\hat{h}'\hat{h}'\rrbracket\Big]\hat{r} \tag{3.180}
$$

$$
\begin{aligned}
\overline{\mathcal{L}}_{A:hhrr} = {} & -\frac{1}{12}(\partial\hat{h})_\mu(\partial^\mu\hat{h})\hat{r}^2 + \frac{1}{24}(\partial_\mu\hat{h})^2\hat{r}^2 - \frac{1}{24}(\partial_\mu\hat{h}_{\nu\rho})^2\hat{r}^2 + \frac{1}{12}(\partial_\mu\hat{h}_{\nu\rho})(\partial^\nu\hat{h}^{\mu\rho})\hat{r}^2 \\
& - \frac{1}{6}\hat{h}(\partial\hat{h})_\mu\hat{r}(\partial^\mu\hat{r}) + \frac{1}{3}\hat{h}_{\mu\nu}(\partial\hat{h})^\mu\hat{r}(\partial^\nu\hat{r}) + \frac{1}{6}\hat{h}(\partial_\mu\hat{h})\hat{r}(\partial^\mu\hat{r}) - \frac{1}{3}\hat{h}_{\mu\nu}(\partial_\rho\hat{h}^{\mu\nu})(\partial^\rho\hat{h}) \\
& - \frac{1}{3}\hat{h}_{\mu\nu}(\partial^\mu\hat{h})\hat{r}(\partial^\nu\hat{r}) + \frac{1}{3}\hat{h}_{\nu\rho}(\partial^\rho\hat{h}^{\mu\nu})\hat{r}(\partial_\mu\hat{r}) + \frac{1}{16}\hat{h}^2(\partial_\mu\hat{r})^2 - \frac{1}{8}\llbracket\hat{h}\hat{h}\rrbracket(\partial_\mu\hat{r})^2 \\
& - \frac{1}{4}\hat{h}\hat{h}_{\mu\nu}(\partial^\mu\hat{r})(\partial^\nu\hat{r}) + \frac{1}{2}\llbracket\hat{h}\hat{h}\rrbracket_{\mu\nu}(\partial^\mu\hat{r})(\partial^\nu\hat{r})
\end{aligned}
\tag{3.181}
$$

$$
\overline{\mathcal{L}}_{B:hhrr} = \frac{5}{12}\Big[\llbracket\hat{h}'\rrbracket^2 - \llbracket\hat{h}'\hat{h}'\rrbracket\Big]\hat{r}^2 \tag{3.182}
$$

$$\overline{\mathcal{L}}_{A:hrrr} = \frac{1}{6\sqrt{6}}\left[2(\partial\hat{h})_\mu \hat{r}^2(\partial^\mu\hat{r}) - 2(\partial_\mu\hat{h})\hat{r}^2(\partial^\mu\hat{r}) - 3\hat{h}\hat{r}(\partial_\mu\hat{r})^2 + 6\hat{h}_{\mu\nu}\hat{r}(\partial^\mu\hat{r})(\partial^\nu\hat{r})\right] \quad (3.183)$$

$$\overline{\mathcal{L}}_{B:hrrr} = 0 \quad (3.184)$$

$$\overline{\mathcal{L}}_{A:rrrr} = \frac{1}{8}\hat{r}^2(\partial_\mu\hat{r})^2 \quad (3.185)$$

$$\overline{\mathcal{L}}_{B:rrrr} = 0 \quad (3.186)$$

3.5 Appendix: WFE Expressions

This appendix derives formulas for weak field expanding the inverse metric $\tilde{G}^{MN}$ and the invariant volume element $\sqrt{|\det G|}$.

3.5.1 Inverse Metric

Consider a metric G on X-dimensional spacetime of the form

$$G_{MN} \equiv c_0\eta_{MN} + H_{MN} \quad (3.187)$$

where the real number c_0 is positive. If H_{MN} is small relative to $c_0\eta_{MN}$, we may weak field expand $\tilde{G}^{MN}$ in the field H_{MN}. Because G_{MN} only depends on H_{MN}, the form of this expansion must be, using the twice-squared bracket notation defined in Section 2.2.1,

$$\tilde{G}^{MN} \equiv \sum_{n=0}^{+\infty} \tilde{c}_n \, [\![H^n]\!]^{MN} \quad (3.188)$$

We can solve for the unknown coefficients $\tilde{c}$ in Eq. (3.188) by imposing the inversion condition $G_{MN}\tilde{G}^{NP} = \eta_M^P$ like so:

$$\eta_M^P \equiv \left[c_0\eta_{MN} + H_{MN}\right]\left[\sum_{n=0}^{+\infty} \tilde{c}_n \, [\![H^n]\!]^{NP}\right] \quad (3.189)$$

$$= c_0 \sum_{n=0}^{+\infty} \tilde{c}_n [\![H^n]\!]_M^P + \sum_{n=0}^{+\infty} \tilde{c}_n [\![H^{n+1}]\!]_M^P \quad (3.190)$$

$$= c_0\tilde{c}_0 \, \eta_M^P + \sum_{n=1}^{+\infty}(c_0\tilde{c}_n + \tilde{c}_{n-1})[\![H^n]\!]_M^P \quad (3.191)$$

which forces the recursive relations

$$\tilde{c}_0 = c_0^{-1} \qquad \tilde{c}_n = -c_0^{-1}\tilde{c}_{n-1} = (-1)^n c_0^{-(n+1)} \quad (3.192)$$

such that, when $G_{MN} = c_0\eta_M + H_{MN}$,

$$\tilde{G}^{MN} = \sum_{n=0}^{+\infty}(-1)^n c_0^{-(n+1)}[\![H^n]\!]^{MN} \quad (3.193)$$

3.5.2 Covariant Volume Factor

Next, let us weak field expand the covariant spacetime volume factor $\sqrt{|\det G|}$ for various choices of the metric G_{MN}. As before, $\det G$ here refers to the determinant of the matrix of components G_{MN}. We will increase the complexity of G in stages until it is of the form of the RS1 metric.

3.5.2.1 Flat Minkowski Spacetime

Define $\eta_{MN} \equiv \mathrm{Diag}(+1,-1,\ldots,-1) = \eta^{MN}$ as the flat X-dimensional spacetime metric. Then, immediately,

$$\sqrt{|\det\eta|} = \sqrt{|(+1)\cdot(-1)^{X-1}|} = 1 \tag{3.194}$$

To prepare for more complicated cases, let us also calculate this another way. Namely, we may use the formula,

$$\det A = \exp\{\mathrm{tr}\,[\mathrm{Log}\,(A)]\} \tag{3.195}$$

to write

$$\sqrt{\pm\det A} = \begin{cases} \left|\exp\left[\frac{1}{2}\mathrm{tr}[\mathrm{Log}\,(A)]\right]\right| \\ \left|i\exp\left\{\frac{1}{2}\mathrm{tr}[\mathrm{Log}\,(A)]\right\}\right| \end{cases} \tag{3.196}$$

To ensure the LHS equals $\sqrt{|\det A|}$ when applied to $A=\eta$ (and, later, $A=G$), we will take the $+$ case when X is odd and $-$ case when X is even. This allows us to write,

$$\sqrt{|\det A|} = \left|i^{X+1}\exp\left\{\frac{1}{2}\mathrm{tr}[\mathrm{Log}\,(A)]\right\}\right| \tag{3.197}$$

The matrix logarithm present on the RHS of Eq. (3.197) is defined via power series,

$$\mathrm{Log}\,(\mathbb{1}+A) \equiv \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}A^n = A - \frac{1}{2}A^2 + \frac{1}{3}A^3 - \ldots \tag{3.198}$$

For a diagonal matrix (and using principal values),

$$\mathrm{Log}\,[\mathrm{Diag}\,(A_1,\ldots,A_N)] = \mathrm{Diag}\,[\log(A_1),\ldots,\log(A_N)] \tag{3.199}$$

Hence,

$$\mathrm{Log}\,\eta = \mathrm{Diag}\,[\log(+1),\log(-1),\ldots,\log(-1)] \tag{3.200}$$
$$= \mathrm{Diag}\,(0,i\pi,i\pi,\cdots,i\pi) \tag{3.201}$$

and so

$$\exp\left[\frac{1}{2}\mathrm{tr}\,(\mathrm{Log}\,\eta)\right] = \exp\left[\frac{1}{2}(X-1)i\pi\right] = i^{(X-1)} \tag{3.202}$$

such that,

$$\sqrt{|\det \eta|} = \left| i^{2X} \right| = \left| (-1)^X \right| = 1 \tag{3.203}$$

which is consistent with our first calculation. This second method is excessive for the flat metric. However, it is useful for more complicated metrics whose determinants cannot be calculated directly.

3.5.2.2 Perturbing Minkowski Spacetime

Next, we consider the perturbed metric $G_{MN} \equiv c_0 \eta_{MN} + H_{MN}$ (note this is the metric we used in the previous subsection). Our goal is to weak field expand $\sqrt{|\det G|}$, i.e. calculate $\sqrt{|\det G|}$ as perturbative expansion in H near the background metric η. Because $\eta = \tilde{\eta}$ and $\eta^2 = \eta\tilde{\eta} = \mathbb{1}$ when considered as matrices, we can write G as the following product:

$$G = c_0 \eta + H \qquad = \eta(c_0 \mathbb{1} + \eta H) \tag{3.204}$$

If $[\overline{A}, \overline{B}] = 0$ for matrices $\overline{A}$ and $\overline{B}$, then we can apply $\text{Log}(\overline{AB}) = \text{Log}(\overline{A}) + \text{Log}(\overline{B})$. However, this is not the case for the product in the above expression: $\eta\eta H = H = [H_{MN}]$ whereas $\eta H \eta = [H^{MN}]$ such that $[\eta, c_0\mathbb{1} + \eta H] = [\eta, \eta H]$ is nonzero. Thankfully, there's a simplification afforded to us by the Baker-Campbell-Hausdorff (BCH) formula. The BCH formula is of the form,

$$\text{Exp}(\overline{A})\,\text{Exp}(\overline{B}) = \text{Exp}\left(\overline{A} + \overline{B} + \tfrac{1}{2}[\overline{A}, \overline{B}] + \dots\right) \tag{3.205}$$

This is useful to us after making the replacements $(\overline{A}, \overline{B}) \to (\text{Log}\,\overline{A}, \text{Log}\,\overline{B})$ and taking the matrix logarithm of both sides. Then the BCH becomes

$$\text{Log}(\overline{AB}) = \text{Log}(\overline{A}) + \text{Log}(\overline{B}) + \tfrac{1}{2}\left[\text{Log}(\overline{A}), \text{Log}(\overline{B})\right] + \dots \tag{3.206}$$

We intend to take the trace of both sides to apply the determinant formula Eq. (3.195), and (thankfully) the trace distributes over addition:

$$\text{tr}\,\text{Log}(\overline{AB}) = \text{tr}\,\text{Log}(\overline{A}) + \text{tr}\,\text{Log}(\overline{B}) + \tfrac{1}{2}\text{tr}\left[\text{Log}(\overline{A}), \text{Log}(\overline{B})\right] + \dots \tag{3.207}$$

where higher-order terms contain traces of increasingly-many commutators. But the trace of any commutator vanishes because $\text{tr}(XY) = \text{tr}(YX)$, such that

$$\text{tr}[X, Y] = \text{tr}(XY) - \text{tr}(YX) = \text{tr}(XY) - \text{tr}(XY) = 0 \tag{3.208}$$

Therefore, the traces of all commutators in our modified BCH formula vanish, such that

$$\text{tr}\,\text{Log}(\overline{AB}) = \text{tr}\,\text{Log}(\overline{A}) + \text{tr}\,\text{Log}(\overline{B}) \tag{3.209}$$

which implies

$$\sqrt{|\det \overline{AB}|} = \left| i^{X+1} \exp\left\{\tfrac{1}{2}\text{tr}[\text{Log}\,(\overline{A})]\right\} \exp\left\{\tfrac{1}{2}\text{tr}[\text{Log}\,(\overline{B})]\right\} \right| \tag{3.210}$$

and, setting $\overline{A} = \eta$ and $\overline{B} = c_0 \mathbb{1} + \eta H$,

$$\sqrt{|\det G|} = \left| i^{X+1} \exp\left\{\tfrac{1}{2}\mathrm{tr}[\mathrm{Log}\,(c_0\eta)]\right\} \exp\left\{\tfrac{1}{2}\mathrm{tr}[\mathrm{Log}\,(\mathbb{1} + \eta H)]\right\} \right| \tag{3.211}$$

The first exponential can be evaluated exactly. Because

$$\begin{aligned} \mathrm{tr}[\mathrm{Log}(c_0\eta)] &= \log[\det(c_0\eta)] && (3.212)\\ &= \log\left[c_0^X(-1)^{X-1}\right] && (3.213)\\ &= \log(c_0^X) + (X-1)i\pi && (3.214) \end{aligned}$$

it is the case that

$$\exp\left\{\tfrac{1}{2}\mathrm{tr}\left[\mathrm{Log}\,(c_0\eta))\right]\right\} = c_0^{X/2}\exp\left[\tfrac{1}{2}(X-1)i\pi\right] = i^{X-1}c_0^X \tag{3.215}$$

Substituting this into Eq. (3.211), we obtain the exact expression

$$\sqrt{|\det G|} = c_0^{X/2}\exp\left\{\tfrac{1}{2}\mathrm{tr}\left[\mathrm{Log}\,(\mathbb{1} + \eta H)\right]\right\} \tag{3.216}$$

Finally, using the perturbative expression for the matrix logarithm Eq. (3.198), we obtain

$$\sqrt{|\det(c_0\eta + H)|} = c_0^{X/2}\prod_{n=1}^{+\infty}\exp\left(\frac{(-1)^{n-1}}{2n}[\![H^n]\!]\right) \tag{3.217}$$

where $[\![H]\!] = \eta^{MN}\,H_{NM}$, $[\![H^2]\!] = \eta^{MN}\,H_{NP}\,\eta^{PQ}\,\eta_{QM}$, and so-on. To obtain the $\mathcal{O}(H^n)$ terms in $\sqrt{|\det G|}$, we should expand each exponential in the product to $\mathcal{O}(H^n)$. For example, to obtain $\mathcal{O}(H^4)$ results, the relevant exponentials and their expansions are

$$\exp\left(+\tfrac{1}{2}[\![H]\!]\right) = 1 + \tfrac{1}{2}[\![H]\!] + \tfrac{1}{8}[\![H]\!]^2 + \tfrac{1}{48}[\![H]\!]^3 + \tfrac{1}{384}[\![H]\!]^4 + \mathcal{O}(H^5) \tag{3.218}$$

$$\exp\left(-\tfrac{1}{4}[\![H^2]\!]\right) = 1 - \tfrac{1}{4}[\![H^2]\!] + \tfrac{1}{32}[\![H^2]\!]^2 + \mathcal{O}(H^6) \tag{3.219}$$

$$\exp\left(+\tfrac{1}{8}[\![H^3]\!]\right) = 1 + \tfrac{1}{8}[\![H^3]\!] + \mathcal{O}(H^6) \tag{3.220}$$

$$\exp\left(-\tfrac{1}{16}[\![H^4]\!]\right) = 1 - \tfrac{1}{16}[\![H^4]\!] + \mathcal{O}(H^8) \tag{3.221}$$

which yields

$$\begin{aligned} \sqrt{|\det G|} = c_0^{X/2}\Bigg[&1 + \frac{1}{2}[\![H]\!] + \frac{1}{8}\Big([\![H]\!]^2 - 2[\![H^2]\!]\Big)\\ &+ \frac{1}{48}\Big([\![H]\!]^3 - 6[\![H]\!]\,[\![H^2]\!] + 6[\![H^3]\!]\Big)\\ &+ \frac{1}{384}\Big([\![H]\!]^4 - 12[\![H]\!]^2\,[\![H^2]\!] + 12[\![H^2]\!]^2 + 24[\![H]\!]\,[\![H^3]\!] - 24[\![H^4]\!]\Big) + \mathcal{O}(H^5)\Bigg] \end{aligned} \tag{3.222}$$

3.5.2.3 Block Diagonal Extension

Suppose we expand the metric G even further into an $(X+1)$-dimensional object $\overline{G}$, so that

$$\overline{G} = \begin{pmatrix} w_0(c_0\eta + H) & \vec{0}^T \\ \vec{0} & -v_0^2 \end{pmatrix} \tag{3.223}$$

where w_0 and v_0 are real and positive. To find $\sqrt{|\det \overline{G}|}$, we employ a fact about block diagonal matrices. Let M be a block diagonal matrix $M \equiv \mathrm{Diag}(A, B)$ where A and B are square matrices. We may define additional matrices $A' \equiv \mathrm{Diag}(A, \mathbb{1}_B)$ and $B' \equiv \mathrm{Diag}(\mathbb{1}_A, B)$, where $\mathbb{1}_A$ and $\mathbb{1}_B$ are identity matrices of the same dimensionality as A and B respectively. $\overline{A}$ and $\overline{B}$ commute ($[A', B'] = 0$) their products recovers M ($M = A'B'$). Thus, the BCH formula implies $\mathrm{Log}(M) = \mathrm{Log}(A') + \mathrm{Log}(B')$, and

$$\begin{aligned} \det(M) &= \exp\left[\mathrm{tr}(\mathrm{Log}\, M)\right] && (3.224) \\ &= \exp\left[\mathrm{tr}\left(\mathrm{Log}\, A' + \mathrm{Log}\, B'\right)\right] && (3.225) \\ &= \exp\left[\mathrm{tr}(\mathrm{Log}\, A') + \mathrm{tr}(\mathrm{Log}\, B')\right] && (3.226) \\ &= \exp\left[\mathrm{tr}(\mathrm{Log}\, A')\right] \, \exp\left[\mathrm{tr}(\mathrm{Log}\, B')\right] && (3.227) \\ &= \det(A') \, \det(B') && (3.228) \end{aligned}$$

Because $\det(\mathbb{1}_A) = \det(\mathbb{1}_B) = 1$, this result implies $\det(A') = \det(A)\det(\mathbb{1}_A) = \det(A)$ and $\det(B') = \det(B)\det(\mathbb{1}_B) = \det(B)$, such that

$$\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \det(A) \, \det(B) \tag{3.229}$$

This generalizes to multiple blocks via recursion, i.e. the determinant of a block diagonal matrix $\det[\mathrm{Diag}(M_1, M_2, \ldots, M_n)]$ is the product of the determinant of the individual blocks $\det(M_1) \, \det(M_2) \, \ldots \, \det(M_n)$. Using this on our extended metric, we find

$$\sqrt{|\det G|} = \sqrt{\left|\det[w_0(c_0\eta + H)] \, \det(-v_0^2)\right|} = v_0 w_0^{X/2} \sqrt{|\det(c_0\eta + H)|} \tag{3.230}$$

from which we can use the previous perturbative result, Eq. (3.217). This is the form relevant to the 5D RS1 model.

Chapter 4

The 4D Effective RS1 Model and its Sum Rules

4.1 Chapter Summary

The principle result of the last chapter was the weak field expansion (WFE) of the 5D RS1 Lagrangian, as summarized in Eqs. (3.164)-(3.186). Up to quartic order in the fields, we derived each term $\mathcal{L}^{(\text{RS})}_{h^H r^R}$ containing H instances of the field $\hat{h}_{\mu\nu}(x,y)$ and R instances of the field $\hat{r}(x)$, and partitioned them into A-type and B-type terms according to Eq. (3.164):

$$\mathcal{L}^{(\text{RS})}_{h^H r^R} = \kappa^{H+R-2}\left[e^{-\pi k r_c}\varepsilon^{+2}\right]^R \left[\varepsilon^{-2}\overline{\mathcal{L}}_{A:h^H r^R} + \varepsilon^{-4}\overline{\mathcal{L}}_{B:h^H r^R}\right] \tag{4.1}$$

This chapter demonstrates how the 5D fields $\hat{h}_{\mu\nu}(x,y)$ and $\hat{r}(x)$ in the 5D WFE RS1 Lagrangian encode information about 4D spin-2 and spin-0 fields respectively. For example, consider the quadratic terms obtained via this process, as recorded in Eqs. (3.165)-(3.168),

$$\mathcal{L}^{(\text{RS})}_{hh} = \varepsilon^{-2}\left[(\partial\hat{h})_\mu(\partial^\mu\hat{h}) - (\partial\hat{h})^2_\mu + \frac{1}{2}(\partial_\mu\hat{h}_{\nu\rho})^2 - \frac{1}{2}(\partial_\mu\hat{h})^2\right] + \varepsilon^{-4}\left[\frac{1}{2}[\![\hat{h}']\!]^2 - \frac{1}{2}[\![\hat{h}'\hat{h}']\!]\right] \tag{4.2}$$

$$\mathcal{L}^{(\text{RS})}_{rr} = \left[e^{-\pi k r_c}\varepsilon^{+2}\right]^2 \cdot \left[\frac{1}{2}(\partial_\mu\hat{r})^2\right] \tag{4.3}$$

These are structurally similar to the 4D Lagrangians from Eqs. (2.345), (2.346), and (2.352):

$$\mathcal{L}^{(s=2)}_{\text{massless}} \equiv (\partial\hat{h})_\mu(\partial^\mu\hat{h}) - (\partial\hat{h})^2_\mu + \frac{1}{2}(\partial_\mu\hat{h}_{\nu\rho})^2 - \frac{1}{2}(\partial_\mu\hat{h})^2 \tag{4.4}$$

$$\mathcal{L}^{(s=2)}_{\text{massive}} \equiv \mathcal{L}^{(s=2)}_{\text{massless}} + m^2\left[\frac{1}{2}\hat{h}^2 - \frac{1}{2}[\![\hat{h}\hat{h}]\!]\right] \tag{4.5}$$

$$\mathcal{L}^{(s=0)}_{\text{massless}} \equiv \frac{1}{2}(\partial_\mu\hat{r})^2 \tag{4.6}$$

which are the canonical massless spin-2, massive spin-2, and massless spin-0 Lagrangians respectively. Specifically, if $\hat{h}_{\mu\nu}(x,y)$ is momentarily assumed y-independent, then the terms proportional to ε^{-4} in $\mathcal{L}^{(\text{RS})}_{hh}$ from Eq. (4.2) vanish. The remaining terms are proportional to ε^{-2} and exactly mimic the Lorentz structures of the massless spin-2 Lagrangian (Eq. (4.4)).

Furthermore, if we restore the y-dependence of $\hat{h}_{\mu\nu}(x,y)$, the Lorentz structures of the newly-revived ε^{-4} terms mimic the Fierz-Pauli mass terms of the massive spin-2 Lagrangian (Eq. (4.5)). This hints (correctly) that the 5D field $\hat{h}_{\mu\nu}(x,y)$ contains information about 4D spin-2 particle excitations, with its y-dependence specifically encoding information about 4D particle masses. Meanwhile, the Lorentz structure of $\mathcal{L}_{rr}^{(\text{RS})}$ in Eq. (4.3) directly mimics the massless spin-0 Lagrangian (Eq. (4.6)). The absence of a massive spin-0 structure for the y-independent $\hat{r}(x)$ field synergizes well with our existing observation that massive spin-2 structures arose from the y-dependence of $h_{\mu\nu}(x,y)$. In all, the y-independent field $\hat{r}$ seemingly contains information about a massless spin-0 particle excitation.

This chapter formalizes how the 5D fields $\hat{h}_{\mu\nu}(x,y)$ and $\hat{r}(x)$ generate 4D fields and thus 4D particle content. The key technique is Kaluza-Klein (KK) decomposition, which allows the 5D fields to be written as sums of 4D fields weighted by extra-dimensional wavefunctions, e.g.

$$\hat{h}_{\mu\nu}(x,y) = \frac{1}{\sqrt{\pi r_c}} \sum_{n=0}^{+\infty} \hat{h}_{\mu\nu}^{(n)}(x)\psi_n(\varphi) \qquad \hat{r}(x) = \frac{1}{\sqrt{\pi r_c}} \hat{r}^{(0)}(x)\psi_0 \tag{4.7}$$

where $\{\psi_n(\varphi)\}$ are the aforementioned wavefunctions and $\varphi = y/r_c \in [-\pi, +\pi]$ parameterizes the extra dimension. The zero mode wavefunction ψ_0 present in both decompositions is independent of φ and thus constant across the extra dimension. The wavefunctions ψ_n solve a Sturm-Liouville (SL) equation, and thereby form a complete basis for orbifolded-even continuous functions $f(\varphi)$:

$$f(\varphi) = \frac{1}{\sqrt{\pi r_c}} f_n\, \psi_n(\varphi) \qquad \Longrightarrow \qquad f_n = \sqrt{\frac{r_c}{\pi}} \int_{-\pi}^{+\pi} d\varphi \quad \varepsilon^{-2} f(\varphi)\psi_n(\varphi) \tag{4.8}$$

where $\varepsilon \equiv \exp(kr_c|\varphi|)$. Although this decomposition appears more symmetric when expressed in terms of $y = \varphi r_c$,

$$f(y) = \frac{1}{\sqrt{\pi r_c}} f_n \; \psi_n\!\left(\frac{y}{r_c}\right) \qquad \Longrightarrow \qquad f_n = \frac{1}{\sqrt{\pi r_c}} \int_{-\pi r_c}^{+\pi r_c} dy \quad \varepsilon^{-2} f(y)\, \psi_n\!\left(\frac{y}{r_c}\right) \tag{4.9}$$

working in terms of φ makes manifest the fact that the wavefunctions and mass spectrum $\{\mu_n\} = \{m_n r_c\}$ depend only on the parameter combination kr_c (as opposed to k and r_c independently). Thus, we favor the use of φ during KK decomposition and the subsequent investigation of important integrals. Such integrals are generated when the KK decomposition ansatz is utilized while determining the 4D effective Lagrangian,

$$\mathcal{L}_{\text{4D}}^{(\text{eff})}(x) \equiv \int_{-\pi r_c}^{+\pi r_c} dy \;\, \mathcal{L}_{\text{5D}}(x,y) \tag{4.10}$$

In this way, the 4D effective theory bundles all extra-dimensional dependence into various integrals of products of wavefunctions.

The rest of this chapter proceeds as follows. Footnotes detail how results in this chapter relate to our published works.

- Section 4.2 introduces KK decomposition and derives the wavefunctions necessary for KK decomposition to yield canonical 4D particle content. Because of its importance for future work, the derivation is performed under slightly more general circumstances than is required for this dissertation.

- Section 4.3 then applies KK decomposition to the quadratic 5D Lagrangians, thereby demonstrating that $\hat{h}_{\mu\nu}(x,y)$ embeds a massless spin-2 field $\hat{h}^{(0)}_{\mu\nu}(x)$ (the graviton) and a tower of massive spin-2 fields $\hat{h}^{(n)}_{\mu\nu}(x)$ (massive KK modes) whereas $\hat{r}(x)$ only embeds a massless spin-0 field $\hat{r}^{(0)}(x)$ (the radion). KK decomposition is then applied to the more general weak field expanded 5D Lagrangian. This requires integrating over the extra dimension, which results in interactions weighted by integrals of products of KK wavefunctions. These integrals define A-type and B-type couplings. The kr_c dependence of these coupling integrals in the large kr_c limit is briefly considered.[1]

- Section 4.4 derives relations (sum rules) between those coupling integrals and the spin-2 KK mode masses.[2]

The results of Sections 4.3 and 4.4 are essential building blocks for the main outcomes of this dissertation. In the next and final chapter of this dissertation, the 4D effective Lagrangian derived in Section 4.3 will be used to calculate scattering amplitudes. The sum rules derived in Section 4.4 will prove vital for ensuring cancellations in the most divergent high-energy growth of those amplitudes.

4.2 Wavefunction Derivation[3]

Let us now elaborate on the connection between 5D and 4D fields that was established in the chapter summary, and in doing so derive explicit expressions for the wavefunctions that will be utilized in the KK decomposition procedure. To demonstrate the KK decomposition is generically possible, we assume a quadratic 5D Lagrangian $\mathcal{L}_{5\text{D}}$ can be decomposed into a sum of quadratic 4D Lagrangians, derive constraints that are necessary for that assumption to hold true, demonstrate all constraints can be satisfied by solving a certain Sturm-Liouville problem, and then reveal we could have used that problem's solution set to begin with. However, rather than work with Eq. (4.2) directly, let us generalize somewhat. This generalization is excessive for our present goals, but is important when considering (for example) natural extensions of this work, including the addition of 5D bulk scalar matter or when constructing models of radion stabilization.

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[1] A-type and B-type couplings were originally defined in [17]. The decomposition and derivation of the 4D effective RS1 Lagrangian was originally published in [18]. The generalized coupling structure $x^{(p)}$ is new to this dissertation, as are the generalizations of the A-type and B-type couplings that it implies.

[2] Most of the elastic sum rules derived in this chapter were originally published in [17] and later proved in [18]; this section significantly generalizes the proofs in [18], and the inelastic results are entirely new to this dissertation.

[3] This section was originally published as Appendix B of [18]. In addition to some changes in wording, certain points have been elaborated on.

Thus, instead of the massless 5D field $\hat{h}_{\mu\nu}(x,y)$, we consider a *massive* 5D field $\Phi_{\vec{\alpha}}(x,y)$ defined over the 5D bulk by a Lagrangian

$$\mathcal{L}_{5\text{D}} = Q_A^{\mu\vec{\alpha}\nu\vec{\beta}} e^{-2k|y|}(\partial_\mu \Phi_{\vec{\alpha}})(\partial_\nu \Phi_{\vec{\beta}}) + Q_B^{\vec{\alpha}\vec{\beta}} \left\{ e^{-4k|y|}(\partial_y \Phi_{\vec{\alpha}})(\partial_y \Phi_{\vec{\beta}}) + m_\Phi^2 e^{-4k|y|}\Phi_{\vec{\alpha}}\Phi_{\vec{\beta}} \right\} \tag{4.11}$$

where the index $\vec{\alpha}$ is a list of Lorentz indices and m_Φ is the 5D mass of the field. The Lorentz tensors $Q_A^{\mu\vec{\alpha}\nu\vec{\beta}}$ and $Q_B^{\vec{\alpha}\vec{\beta}}$ will eventually be chosen to ensure this procedure yields KK modes with canonical kinetic terms. Note that this Lagrangian can be written equivalently as

$$\mathcal{L}_{5\text{D}} \cong Q_A^{\mu\vec{\alpha}\nu\vec{\beta}} e^{-2k|y|}(\partial_\mu \Phi_{\vec{\alpha}})(\partial_\nu \Phi_{\vec{\beta}}) + Q_B^{\vec{\alpha}\vec{\beta}} \left\{ -\Phi_{\vec{\alpha}} \cdot \partial_y \left[e^{-4k|y|}(\partial_y \Phi_{\vec{\beta}}) \right] + m_\Phi^2 e^{-4k|y|}\Phi_{\vec{\alpha}}\Phi_{\vec{\beta}} \right\} \tag{4.12}$$

via integration by parts. By performing a mode expansion (KK decomposition) on Eq. (4.12) according to the ansatz

$$\Phi_{\vec{\alpha}}(x,y) = \frac{1}{\sqrt{\pi r_c}} \sum_{n=0}^{+\infty} \Phi_{\vec{\alpha}}^{(n)}(x)\psi_n(y) \ , \tag{4.13}$$

we obtain

$$\begin{aligned} \mathcal{L}_{5\text{D}} \cong \frac{1}{\pi r_c} \sum_{m,n=0}^{+\infty} & Q_A^{\mu\vec{\alpha}\nu\vec{\beta}} (\partial_\mu \Phi_{\vec{\alpha}}^{(m)})(\partial_\nu \Phi_{\vec{\beta}}^{(n)})\, e^{-2k|y|}\psi^{(m)}\psi^{(n)} \\ & + Q_B^{\vec{\alpha}\vec{\beta}} \Phi_{\vec{\alpha}}^{(m)} \Phi_{\vec{\beta}}^{(n)}\, \psi^{(m)} \left\{ -\partial_y \left[e^{-4k|y|}(\partial_y \psi_n) \right] + m_\Phi^2 e^{-4k|y|}\psi_n \right\} . \end{aligned} \tag{4.14}$$

Integrating over the extra dimension as in Eq. (4.10) then yields the following effective 4D Lagrangian:

$$\mathcal{L}_{4\text{D}}^{(\text{eff})} = \sum_{m,n=0}^{+\infty} Q_A^{\mu\vec{\alpha}\nu\vec{\beta}} (\partial_\mu \Phi_{\vec{\alpha}}^{(m)})(\partial_\nu \Phi_{\vec{\beta}}^{(n)}) \cdot N_A^{(m,n)} + Q_B^{\vec{\alpha}\vec{\beta}} \Phi_{\vec{\alpha}}^{(m)} \Phi_{\vec{\beta}}^{(n)} \cdot N_B^{(m,n)} \ , \tag{4.15}$$

where $N_A^{(m,n)}$ and $N_B^{(m,n)}$ equal

$$N_A^{(m,n)} = \frac{1}{\pi r_c} \int_{-\pi r_c}^{+\pi r_c} dy \;\, e^{-2k|y|}\psi_m \psi_n \ , \tag{4.16}$$

$$N_B^{(m,n)} = \frac{1}{\pi r_c} \int_{-\pi r_c}^{+\pi r_c} dy \;\, \psi_m \left\{ -\partial_y \left[e^{-4k|y|}(\partial_y \psi_n) \right] + m_\Phi^2 e^{-4k|y|}\psi_n \right\} . \tag{4.17}$$

We desire that this process yields a particle spectrum described by canonical 4D Lagrangians for particles of definite spins and masses. Specifically, we desire that a (bosonic) mode field $\phi_{\vec{\alpha}}(x)$ in the KK spectrum is described by a Lagrangian

$$q_A^{\mu\vec{\alpha}\nu\vec{\beta}} (\partial_\mu \phi_{\vec{\alpha}})(\partial_\nu \phi_{\vec{\beta}}) + m^2 q_B^{\vec{\alpha}\vec{\beta}} \phi_{\vec{\alpha}}\phi_{\vec{\beta}} \ , \tag{4.18}$$

where m is the mass of the KK mode, and the quantities q_A and q_B are Lorentz tensor structures that reproduce the canonical quadratic Lagrangian appropriate for the internal spin of $\phi_{\vec{\alpha}}$. For example, a massive spin-2 field $\hat{h}_{\mu\nu}$ has the canonical quadratic Lagrangian Eq. (4.5), such that $\phi_{\vec{\alpha}}(x) = \hat{h}_{\alpha_1\alpha_2}(x)$ and we may choose

$$q_A^{\mu\alpha_1\alpha_2\nu\beta_1\beta_2} = \eta^{\mu\alpha_1}\eta^{\alpha_2\nu}\eta^{\beta_1\beta_2} - \eta^{\mu\nu}\eta^{\alpha_1\alpha_2}\eta^{\beta_1\beta_2} + \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha_1\beta_1}\eta^{\alpha_2\beta_2} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha_1\alpha_2}\eta^{\beta_1\beta_2} \tag{4.19}$$

$$q_B^{\alpha_1\alpha_2\beta_1\beta_2} = \frac{1}{2}\eta^{\alpha_1\alpha_2}\eta^{\beta_1\beta_2} - \frac{1}{2}\eta^{\alpha_1\beta_1}\eta^{\alpha_2\beta_2} \tag{4.20}$$

For a full KK tower, the corresponding canonical quadratic Lagrangian equals (indexing KK number by n),[4]

$$\mathcal{L}_{\text{4D}}^{(\text{eff})} = \sum_{n=0}^{+\infty} q_A^{\mu\vec{\alpha}\nu\vec{\beta}}(\partial_\mu\phi_{\vec{\alpha}}^{(n)})(\partial_\nu\phi_{\vec{\beta}}^{(n)}) + m_n^2 q_B^{\vec{\alpha}\vec{\beta}}\phi_{\vec{\alpha}}^{(n)}\phi_{\vec{\beta}}^{(n)} \ . \tag{4.21}$$

Comparing to Eq. (4.15), one recovers this form for the choices $Q = q$ (i.e. if the 5D quadratic tensor structures mimic the 4D canonical quadratic tensor structures), $\Phi^{(n)} = \phi^{(n)}$, $N_A^{(m,n)} = \delta_{m,n}$, and $N_B^{(m,n)} = m_n^2\delta_{m,n}$. Consider this condition on $N_B^{(m,n)}$ in more detail. Via the defining equation Eq. (4.17), $N_B^{(m,n)} = m_n^2\delta_{m,n}$ implies

$$\frac{1}{\pi r_c}\int_{-\pi r_c}^{+\pi r_c} dy\ \psi^{(m)}\bigg\{ -\partial_y\Big[e^{-4k|y|}(\partial_y\psi_n)\Big] + m_\Phi^2 e^{-4k|y|}\psi_n\bigg\} = m_n^2\delta_{m,n} \ . \tag{4.22}$$

which then becomes, using the condition $N_A^{(m,n)} = \delta_{m,n}$ and Eq. (4.16),

$$\int_{-\pi r_c}^{+\pi r_c} dy\ \psi_m\bigg\{\partial_y\Big[e^{-4k|y|}(\partial_y\psi_n)\Big] + \Big(m_n^2 e^{-2k|y|} - m_\Phi^2 e^{-4k|y|}\Big)\psi_n\bigg\} = 0 \ . \tag{4.23}$$

If the collection of wavefunctions $\{\psi_m\}$ form a complete set, then Eq. (4.23) implies that they are solutions of the following differential equation

$$\partial_y\Big[e^{-4k|y|}(\partial_y\psi_n)\Big] + \Big(m_n^2 e^{-2k|y|} - m_\Phi^2 e^{-4k|y|}\Big)\psi_n = 0 \ , \tag{4.24}$$

or, when expressed in unitless combinations,

$$0 = \partial_\varphi\Big[e^{-4kr_c|\varphi|}(\partial_\varphi\psi_n)\Big] + \Big((m_n r_c)^2 e^{-2kr_c|\varphi|} - (m_\Phi r_c)^2 e^{-4kr_c|\varphi|}\Big)\psi_n \ . \tag{4.25}$$

[4] Restricting the KK decomposition sum to positive KK indices n is inspired by the RS1 model's orbifold symmetry. For example, if we instead considered a (non-orbifolded) torus, we would sum over all integer n, with the sign of n describing the rotational direction of the particle's momentum around the circular extra dimension. From this perspective, imposing an orbifold symmetry causes the $+n$ and $-n$ non-orbifolded states to be combined into an even superposition which we then call the nth KK mode of the orbifolded theory.

In addition to this differential equation, orbifold symmetry requires that the derivatives of the wavefunctions vanish at the orbifold fixed points, i.e. $(\partial_\varphi \psi_n) = 0$ for $\varphi \in \{0, \pi\}$, which provides the problem with boundary conditions. Finding the solution set $\{\psi_n\}$ (and corresponding values of $\{m_n r_c\}$) of Eq. (4.25) under these boundary conditions is precisely a Sturm-Liouville (SL) problem, for which there is guaranteed a discrete (complete) basis of real wavefunctions satisfying

$$\frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi \; e^{-2kr_c|\varphi|}\psi_m \psi_n = N_A^{(m,n)} \equiv \delta_{m,n} \; , \tag{4.26}$$

as required. Hence, by finding wavefunctions ψ_n that solve Eqs. (4.25) and (4.26), we can KK decompose the fields in Eq. (4.11) according to the ansatz and (so long as $Q = q$) obtain a tower of canonical quadratic Lagrangians (4.18) as desired.

Eq. (4.2) is of the form Eq. (4.11) with $m_\Phi = 0$. In general, when the bulk mass m_Φ vanishes, Eq. (4.25) admits a massless solution ($m_n = 0$) which is flat in the extra dimension ($\partial_y \psi_0 = 0$). Therefore, the 5D field $\hat{h}_{\mu\nu}$ gives rise to a massless 4D field $\hat{h}^{(0)}_{\mu\nu}$, which we identify with the usual (4D) graviton. The 5D field $\hat{r}$ yields a massless 4D field $\hat{r}^{(0)}$ which we identify as the radion; however, note that Eq. (4.3) is not of the form Eq. (4.11) because of the additional warp factors introduced alongside $\hat{r}$, and thus its KK decomposition is derived solely from the y-independence of $\hat{r}$.[5] Normalization fixes ψ_0 to equal

$$\psi_0 = \sqrt{\frac{kr_c\pi}{1 - e^{-2kr_c\pi}}} \; . \tag{4.27}$$

When $m_\Phi \neq 0$, this solution does not exist.

By construction, the SL equation combined with Eq. (4.26) implies an additional quadratic integral condition:

$$\frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi \; e^{-4kr_c|\varphi|}\Big[(\partial_\varphi\psi_m)(\partial_\varphi\psi_n) + (m_\Phi r_c)^2 \psi_m\psi_n\Big] = (m_n r_c)^2 \delta_{m,n} \; . \tag{4.28}$$

When $m_\Phi = 0$, this becomes an orthonormality condition on the set $\{\partial_\varphi \psi_n\}$.

The existence of a discrete solution set of wavefunctions is guaranteed by the SL problem. Following the notation and arguments from [28], we now summarize how to find explicit equations for the non-flat wavefunctions in that solution set. Note that

$$\partial_\varphi|\varphi| = \text{sign}(\varphi) \; , \tag{4.29}$$

$$\partial_\varphi^2|\varphi| = 2\big[\delta(\varphi) - \delta(\varphi - \pi)\big] \; , \tag{4.30}$$

such that $(\partial_\varphi|\varphi|)^2 = 1$ and $\partial_\varphi^2|\varphi| = 0$ when $\varphi \neq 0, \pi$. Thus, away from the orbifold fixed points, Eq. (4.25) may be rewritten by defining quantities $z_n = (m_n/k)e^{+kr_c|\varphi|}$ and $f_n = (m_n^2/k^2)\psi_n/z_n^2$, such that

$$z_n^2 \frac{d^2 f_n}{dz_n^2} + z_n \frac{df_n}{dz_n} + \left[z_n^2 - \left(4 + \frac{m_\Phi^2}{k^2}\right)\right] f_n = 0 \; . \tag{4.31}$$

[5]To prevent the radion from contributing to long-range gravitational forces and to ensure the extra-dimensional is stable against quantum fluctuations, we must include interactions which make the physical 4D spin-0 field become massive, as occurs during radion stabilization [28].

When $m_\Phi = 0$, this differential equation is solved by f_n equal to Bessel functions $J_2(z_n)$ or $Y_2(z_n)$. When $m_\Phi \neq 0$, it is instead solved by Bessel functions $J_\nu(z_n)$ and $Y_\nu(z_n)$ where $\nu^2 \equiv 4 + m_\Phi^2/k^2$. Taking a superposition of the appropriate Bessel functions yields a generic solution f_n, which may then be converted back to ψ_n. By imposing the SL boundary conditions at the orbifold fixed points ($\partial_\varphi \psi_n = 0$ for $\varphi \in \{0, \pi\}$), the wavefunctions are found to equal

$$\psi_n = \frac{\varepsilon^2}{N_n}\left[J_\nu\left(\frac{\mu_n \varepsilon}{kr_c}\right) + b_{n\nu}\, Y_\nu\left(\frac{\mu_n \varepsilon}{kr_c}\right)\right] , \tag{4.32}$$

where $\varepsilon \equiv e^{+kr_c|\varphi|}$ and $\mu_n \equiv m_n r_c$, the normalization N_n is determined by Eq. (4.26) (up to a sign that we fix by setting $N_n > 0$ and which yields $\psi_n(0) < 0$ for nonzero n), and the relative weight $b_{n\nu}$ equals

$$b_{n\nu} = -\frac{2J_\nu\Big|_{\mu_n/kr_c} + \dfrac{\mu_n}{kr_c}(\partial J_\nu)\Big|_{\mu_n/kr_c}}{2Y_\nu\Big|_{\mu_n/kr_c} + \dfrac{\mu_n}{kr_c}(\partial Y_\nu)\Big|_{\mu_n/kr_c}} , \tag{4.33}$$

where $\partial J_\nu \equiv \partial J_\nu(z)/\partial z$ and $\partial Y_\nu \equiv \partial Y_\nu(z)/\partial z$. These wavefunctions satisfy Eq. (4.28) where each μ_n solves

$$\begin{aligned}\left[2J_\nu + \frac{\mu_n \varepsilon}{kr_c}(\partial J_\nu)\right]\Bigg|_{\varphi=\pi} \left[2Y_\nu + \frac{\mu_n \varepsilon}{kr_c}(\partial Y_\nu)\right]\Bigg|_{\varphi=0}& \\ - \left[2Y_\nu + \frac{\mu_n \varepsilon}{kr_c}(\partial Y_\nu)\right]\Bigg|_{\varphi=\pi} \left[2J_\nu + \frac{\mu_n \varepsilon}{kr_c}(\partial J_\nu)\right]\Bigg|_{\varphi=0} &= 0 .\end{aligned} \tag{4.34}$$

Although these wavefunctions were derived by solving Eq. (4.25) away from the orbifold fixed points, they solve the equation across the full extra dimension. In particular, they ensure $\partial_\varphi^2 \psi_n = [(m_\Phi r_c)^2 - \varepsilon^2 \mu_n^2]\psi_n$ at $\varphi = 0, \pi$.

Finally, note that given a 5D Lagrangian consistent with Eq. (4.11), the wavefunctions ψ_n and spectrum $\{\mu_n\}$ are entirely determined by the unitless quantities kr_c and $m_\Phi r_c$. In the RS1 model, the 5D field $\hat{h}_{\mu\nu}$ lacks a bulk mass ($m_\Phi = 0$ such that $\nu = 2$), so its KK decomposition is dictated by kr_c alone.

4.3 4D Effective RS1 Model[6]

In this section, we carry out the KK mode expansions of $\hat{h}_{\mu\nu}(x, y)$ and $\hat{r}(x)$, thereby obtaining the 4D particle content of the RS1 model, and discuss the form of the interactions among the 4D fields.

[6] Subsection 4.3.1 was originally published as Subsection III.A of [18]. Subsection 4.3.2 combines content that was originally published as Subsections III.B and C.2 of [18]. Subsection 4.3.3 was originally published as Appendix C of [18]. Notations and terminology have been updated, and paragraphs that describe convenient wavefunction properties and the generalized coupling structure $x^{(p)}$ have been added.

4.3.1 4D Particle Content

The 4D particle content is determined by employing the KK decomposition ansatz [29, 30, 28]:

$$\hat{h}_{\mu\nu}(x,y) = \frac{1}{\sqrt{\pi r_c}} \sum_{n=0}^{+\infty} \hat{h}^{(n)}_{\mu\nu}(x)\psi_n(\varphi) \qquad \hat{r}(x) = \frac{1}{\sqrt{\pi r_c}}\hat{r}^{(0)}(x)\psi_0 \ , \tag{4.35}$$

where we recall that $\varphi = y/r_c$. The operators $\hat{h}^{(n)}_{\mu\nu}$ and $\hat{r}^{(0)}$ are 4D spin-2 and spin-0 fields respectively, while each ψ_n is a wavefunction which solves the following Sturm-Liouville equation

$$\partial_\varphi \left[\varepsilon^{-4}(\partial_\varphi \psi_n)\right] = -\mu_n^2 \varepsilon^{-2}\psi_n \tag{4.36}$$

subject to the boundary condition $(\partial_\varphi\psi_n) = 0$ at $\varphi = 0$ and π, where $\varepsilon \equiv e^{k|y|} = e^{kr_c|\varphi|}$ [28]. As described in the previous section, there exists a unique solution ψ_n (up to normalization) per eigenvalue μ_n, each of which we index with a discrete KK number $n \in \{0, 1, 2, \cdots\}$ such that $\mu_0 = 0 < \mu_1 < \mu_2 < \cdots$. Given a KK number n, the quantity μ_n and wavefunction $\psi_n(\varphi)$ are entirely determined by the value of the unitless nonnegative combination kr_c. We note that with proper normalization the ψ_n satisfy two convenient orthonormality conditions:

$$\frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi \ \varepsilon^{-2}\psi_m\psi_n = \delta_{m,n} \ , \tag{4.37}$$

$$\frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi \ \varepsilon^{-4}(\partial_\varphi\psi_m)(\partial_\varphi\psi_n) = \mu_n^2\delta_{m,n} \ . \tag{4.38}$$

Furthermore, the $\{\psi_n\}$ form a complete set, such that the following completeness relation holds:

$$\delta(\varphi_2 - \varphi_1) = \sum_{j=0}^{+\infty} \frac{1}{\pi}\varepsilon^{-2}\psi_j(\varphi_1)\,\psi_j(\varphi_2) \ . \tag{4.39}$$

Because of the assumptions behind its derivation, the completeness relation can only be used to combine or separate orbifold-even integrands. For example, if $f(\varphi) \neq 0$ is an orbifold-odd function (such as $(\partial_\varphi\psi_n)$), then splitting $f(\varphi)^2$ into a product $f(\varphi)^2 \cdot 1$ is fine,

$$0 < \frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi \quad \varepsilon^{+2} f(\varphi)f(\varphi) = \sum_j \left[\frac{1}{\pi}\int d\varphi \quad f(\varphi)f(\varphi)\psi_j(\varphi)\right]\left[\frac{1}{\pi}\int d\varphi \quad \psi_j(\varphi)\right] \tag{4.40}$$

whereas trying to apply completeness to split $f(\varphi)^2$ as the product $f(\varphi)\cdot f(\varphi)$ yields a contradiction

$$0 < \frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi \quad \varepsilon^{+2} f(\varphi)f(\varphi) \neq \sum_j \left[\frac{1}{\pi}\int d\varphi \quad f(\varphi)\psi_j(\varphi)\right]^2 = 0 \tag{4.41}$$

The completeness relation will be vital to relating different coupling structures present in the 4D effective WFE RS1 Lagrangian.

The KK number $n = 0$ corresponds to $\mu_n = 0$, for which Eq. (4.36) admits a flat wavefunction solution ψ_0 corresponding to the massless 4D graviton. Upon normalization via Eq. (4.37), this wavefunction equals

$$1 = \frac{1}{\pi}\psi_0^2 \int_{-\pi}^{+\pi} \varepsilon^{-2} = \frac{1}{\pi k r_c}\left[1 - e^{-2\pi k r_c}\right] \qquad \Longrightarrow \qquad \psi_0 = \sqrt{\frac{\pi k r_c}{1 - e^{-2\pi k r_c}}} \tag{4.42}$$

up to a phase that we set to $+1$ by convention. This is the wavefunction that Eq. (4.35) associates with the fields $\hat{h}^{(0)}$ and $\hat{r}^{(0)}$. The lack of higher modes in the KK decomposition of $\hat{r}$ reflects its y-independence. In this sense, choosing to associate ψ_0 with $\hat{r}^{(0)}$ in Eq. (4.35) is merely done for convenience.

Before we compute the interactions between 4D states, let us first apply the ansatz to the simpler quadratic terms. This will illustrate how the KK decomposition procedure typically works, as well as why the interaction terms are more complicated. The 5D quadratic $\hat{h}_{\mu\nu}(x, y)$ Lagrangian equals (from Section 3.4)

$$\mathcal{L}^{(\mathrm{RS})}_{hh} = \varepsilon^{-2}\,\overline{\mathcal{L}}_{A:hh} + \varepsilon^{-4}\,\overline{\mathcal{L}}_{B:hh} \;, \tag{4.43}$$

where

$$\overline{\mathcal{L}}_{A:hh} = -\,\hat{h}_{\mu\nu}(\partial^\mu\partial^\nu\hat{h}) + \hat{h}_{\mu\nu}(\partial^\mu\partial_\rho\hat{h}^{\rho\nu}) - \frac{1}{2}\hat{h}_{\mu\nu}(\Box\hat{h}^{\mu\nu}) + \frac{1}{2}\hat{h}(\Box\hat{h}) \;, \tag{4.44}$$

$$\overline{\mathcal{L}}_{B:hh} = -\,\frac{1}{2}[\![\hat{h}'\hat{h}']\!] + \frac{1}{2}[\![\hat{h}']\!]^2 \;, \tag{4.45}$$

A prime indicates differentiation with respect to y and a twice-squared bracket indicates a cyclic contraction of Lorentz indices. Similarly, the quadratic 5D $\hat{r}(x)$ Lagrangian equals,

$$\mathcal{L}^{(\mathrm{RS})}_{rr} = e^{-2\pi k r_c}\,\varepsilon^{+2}\,\overline{\mathcal{L}}_{A:rr} \;, \tag{4.46}$$

where

$$\overline{\mathcal{L}}_{A:rr} = \frac{1}{2}(\partial_\mu\hat{r})(\partial^\mu\hat{r}) \;. \tag{4.47}$$

To obtain the 4D effective equivalents of the above 5D expressions, we must integrate over the extra dimension and employ the KK decomposition ansatz.

First, the quadratic $\hat{h}_{\mu\nu}$ Lagrangian: the first term in Eq. (4.43) becomes

$$\begin{aligned}
\mathcal{L}^{(\mathrm{eff})}_{A:hh} &\equiv \int_{-\pi r_c}^{+\pi r_c} dy\;\varepsilon^{-2}\,\overline{\mathcal{L}}_{A:hh} \\
&= \int_{-\pi r_c}^{+\pi r_c} dy\;\varepsilon^{-2}\left[-\hat{h}_{\mu\nu}(\partial^\mu\partial^\nu\hat{h}) + \hat{h}_{\mu\nu}(\partial^\mu\partial_\rho\hat{h}^{\rho\nu}) - \frac{1}{2}\hat{h}_{\mu\nu}(\Box\hat{h}^{\mu\nu}) + \frac{1}{2}\hat{h}(\Box\hat{h})\right] \\
&= \sum_{m,n=0}^{+\infty}\left[-\hat{h}^{(m)}_{\mu\nu}(\partial^\mu\partial^\nu\hat{h}^{(n)}) + \hat{h}^{(m)}_{\mu\nu}(\partial^\mu\partial_\rho\hat{h}^{(n)\rho\nu}) - \frac{1}{2}\hat{h}^{(m)}_{\mu\nu}(\Box\hat{h}^{(n)\mu\nu}) + \frac{1}{2}\hat{h}^{(m)}(\Box\hat{h}^{(n)})\right] \\
&\qquad \times \frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi\;\varepsilon^{-2}\psi_m\psi_n \;,
\end{aligned} \tag{4.48}$$

whereas its second term becomes

$$\begin{aligned}\mathcal{L}^{(\text{eff})}_{B:hh} &\equiv \int_{-\pi r_c}^{+\pi r_c} dy\ \varepsilon^{-4}\,\overline{\mathcal{L}}_{B:hh}\\ &= \int_{-\pi r_c}^{+\pi r_c} dy\ \varepsilon^{-4}\left[-\frac{1}{2}[\![\hat{h}'\hat{h}']\!] + \frac{1}{2}[\![\hat{h}']\!]^2\right]\\ &= \sum_{m,n=0}^{+\infty}\left[-\frac{1}{2}[\![\hat{h}^{(m)}\hat{h}^{(n)}]\!] + \frac{1}{2}[\![\hat{h}^{(m)}]\!][\![\hat{h}^{(n)}]\!]\right]\frac{1}{\pi r_c^2}\int_{-\pi}^{+\pi} d\varphi\ \varepsilon^{-4}(\partial_\varphi\psi_m)(\partial_\varphi\psi_n)\ . \end{aligned} \tag{4.49}$$

These are simplified via the orthonormality relations Eqs. (4.37) and (4.38), such that the 4D effective Lagrangian resulting from $\mathcal{L}^{(\text{RS})}_{hh}$ equals, using Eqs. (4.4) and (4.5),

$$\begin{aligned}\mathcal{L}^{(\text{RS,eff})}_{hh} &= \mathcal{L}^{(\text{eff})}_{A:hh} + \mathcal{L}^{(\text{eff})}_{B:hh}\\ &= \mathcal{L}^{(s=2)}_{\text{massless}}(\hat{h}^{(0)}) + \sum_{n=1}^{+\infty}\mathcal{L}^{(s=2)}_{\text{massive}}(m_n, \hat{h}^{(n)})\ ,\end{aligned} \tag{4.50}$$

wherein $m_n \equiv \mu_n/r_c$. Therefore, KK decomposition of the 5D field $\hat{h}_{\mu\nu}$ results in the following 4D particle content: a single massless spin-2 mode $\hat{h}^{(0)}$, and countably many massive spin-2 modes $\hat{h}^{(n)}$ with $n \in \{1, 2, \cdots\}$ (each having a corresponding Fierz-Pauli mass term). The zero mode $\hat{h}^{(0)}$ is consistent with the usual 4D graviton, and will be identified as such. The 4D graviton has dimensionful coupling constant $\kappa_{4\text{D}} = 2/M_{\text{Pl}} = \psi_0\kappa/\sqrt{\pi r_c}$ where M_{Pl} is the reduced 4D Planck mass. In terms of the reduced 4D Planck mass, the full 4D Planck mass equals $\sqrt{8\pi}M_{\text{Pl}}$.

Meanwhile, the 4D effective equivalent of $\mathcal{L}^{(\text{RS})}_{rr}$ from Eq. (4.46) equals, using Eq. (4.6),

$$\begin{aligned}\mathcal{L}^{(\text{RS,eff})}_{rr} &= \int_{-\pi r_c}^{+\pi r_c} dy\ \mathcal{L}^{(\text{RS})}_{rr}\\ &= \int_{-\pi r_c}^{+\pi r_c} dy\ e^{-2\pi k r_c}\,\varepsilon^{+2}\left[\frac{1}{2}(\partial_\mu\hat{r})(\partial^\mu\hat{r})\right]\\ &= \frac{1}{2}(\partial_\mu\hat{r}^{(0)})(\partial^\mu\hat{r}^{(0)})\cdot\frac{{\psi_0}^2}{\pi r_c}\int_{-\pi r_c}^{+\pi r_c} dy\ e^{+2k(|y|-\pi r_c)}\\ &= \mathcal{L}^{S=0}_{\text{massless}}(\hat{r}^{(0)})\ .\end{aligned} \tag{4.51}$$

Therefore, KK decomposing the 5D $\hat{r}$ field yields only a single massless spin-0 mode $\hat{r}^{(0)}$, which is called the radion. Note the exponential factor in Eq. (4.46) is inconsistent with the orthonormality equation (4.37), so we had to calculate the integral explicitly. Thankfully, the y-independent radion must possess a flat extra-dimensional wavefunction and so the exponential factor can at most affect its normalization. This would not be the case if the radion's y-dependence could not be gauged away in Subsection 3.3.2.

The RS1 model has three independent parameters according to the above construction: the extra-dimensional radius r_c, the warping parameter k, and the 5D coupling strength κ.

However, we use a more convenient set of independent parameters in practice: the unitless extra-dimensional combination kr_c, the mass m_1 of the first massive KK mode $\hat{h}^{(1)}$, and the reduced 4D Planck mass M_{Pl}. These sets are related according to the following relations:

$$m_1 \equiv \frac{1}{r_c}\mu_1(kr_c) \quad \text{via Eq. (4.36)}\ , \tag{4.52}$$

$$M_{\text{Pl}} \equiv \frac{2}{\kappa\sqrt{k}}\sqrt{1-e^{-2kr_c\pi}}\ . \tag{4.53}$$

In our numerical analysis, we will choose $kr_c \in [0, 10]$, $m_1 = 1$ TeV, and $M_{\text{Pl}} = 2.435 \times 10^{15}$ TeV.

Deriving the quadratic terms proceeded so cleanly in part because all wavefunctions with a nonzero KK number occur in pairs and are thus subject to orthonormality relations. Such simplifications are seldom possible when dealing with a product of three or more 5D $\hat{h}_{\mu\nu}$ fields, and instead the integrals must be dealt with explicitly. Consequently, the RS1 model possesses many nonzero triple couplings and calculating a matrix element for 2-to-2 scattering of massive KK modes typically requires a sum over infinitely many diagrams, each of which is mediated by a different massive KK mode and contains various products of these overlap integrals. The next section details the 4D effective Lagrangian and the origin of those integrals. The final section details relations between KK mode masses and the integrals.

4.3.2 General Procedure

The WFE RS1 Lagrangian equals a sum of terms, wherein each term contains some number of 5D fields and exactly two derivatives. Each derivative in the pair is either a 4D spatial derivative ∂_μ or an extra-dimension derivative ∂_y, and each field is either an $\hat{r}$ or an $\hat{h}_{\mu\nu}$ field. Because the Lagrangian requires an even number of Lorentz indices in order to form a Lorentz scalar, each derivative pair must consist of two copies of the same kind of derivative, i.e. each term in $\mathcal{L}_{\text{5D}}^{(\text{RS})}$ can be classified into one of two categories:

- **A-Type:** The term has two spatial derivatives $\partial_\mu \cdot \partial_\nu$; or
- **B-Type:** The term has two extra-dimensional derivatives $\partial_y \cdot \partial_y$.

In addition to fields and derivatives, every term in $\mathcal{L}_{\text{5D}}^{(\text{RS})}$ has an exponential prefactor. That exponential's specific form is entirely determined by its type (whether A- or B-type) and the number of instances of $\hat{r}$ in the term. Each A-type term is associated with a factor $\varepsilon^{-2} = e^{-2kr_c|\varphi|}$ whereas each B-type term is associated with a factor $\varepsilon^{-4} = e^{-4kr_c|\varphi|}$, and every instance of a radion field provides an additional $e^{-\pi kr_c}\varepsilon^{+2}$ factor. These assignments correctly reproduce the prefactors of Section 3.4.

Consider a generic A-type term with H instances of $\hat{h}$ and R instances of $\hat{r}$. Schematically, it will be of the form,

$$\begin{aligned} X_A &\equiv \kappa^{(H+R-2)} \left[\varepsilon^{-2}\right] \left[e^{-\pi kr_c}\varepsilon^{+2}\right]^R (\partial_\mu^2, \hat{h}^H, \hat{r}^R) \\ &= \kappa^{(H+R-2)}\, e^{-R\pi kr_c}\, \varepsilon^{2(R-1)}\, \overline{X}_A\ , \end{aligned} \tag{4.54}$$

where the combination $\overline{X}_A \equiv (\partial_\mu^2, \hat{h}^H, \hat{r}^R)$ refers to a fully contracted product of two 4D derivatives, H gravitons, and R radions. The μ label on ∂_μ^2 above is only schematic and not literal. Similarly, an equivalent B-type term would be of the form,

$$\begin{aligned} X_B &\equiv \kappa^{(H+R-2)} \left[\varepsilon^{-4}\right] \left[e^{-\pi k r_c}\, \varepsilon^{+2}\right]^R (\partial_y^2, \hat{h}^H, \hat{r}^R) \\ &= \kappa^{(H+R-2)}\, e^{-R\pi k r_c}\, \varepsilon^{2(R-2)}\, \overline{X}_B \ , \end{aligned} \tag{4.55}$$

where the combination $\overline{X}_B \equiv (\partial_y^2, \hat{h}^H, \hat{r}^R)$ refers to a fully contracted product of two extra-dimensional derivatives, H instances of $\hat{h}_{\mu\nu}$, and R instances of $\hat{r}$. By construction, each B-type term we consider never has both of its ∂_y derivatives acting on the same field (i.e. any instances of $\partial_y^2 \hat{h}$ in our 5D Lagrangian have been removed via integration by parts), and so we assume $\overline{X}_B$ also satisfies this property.

We form a 4D effective Lagrangian by first KK decomposing our 5D fields into states of definite mass (Eq. (4.35)) and then integrating over the extra dimension (Eq. (4.10)). For the schematic A-type term, this procedure yields,

$$\begin{aligned} X_A^{(\text{eff})} = \frac{r_c}{(\pi r_c)^{(H+R)/2}} \kappa^{(H+R-2)} &\sum_{n_1,\cdots,n_H=0}^{+\infty} \left(\partial_\mu^2,\ \hat{h}^{(n_1)} \cdots \hat{h}^{(n_H)},\ \left[\hat{r}^{(0)}\right]^R\right) \\ &\times e^{-R\pi k r_c} \int_{-\pi}^{+\pi} d\varphi\ \varepsilon^{2(R-1)} \psi_{n_1} \cdots \psi_{n_H} \left[\psi_0\right]^R \ . \end{aligned}$$

Define a unitless combination a that contains the extra-dimensional overlap integral:

$$a_{(R|\vec{n})} \equiv a_{r\cdots r n_1 \cdots n_H} \equiv \frac{1}{\pi} e^{-R\pi k r_c} \int_{-\pi}^{+\pi} d\varphi\ \varepsilon^{2(R-1)} \psi_{n_1} \cdots \psi_{n_H} \left[\psi_0\right]^R \ , \tag{4.56}$$

where $\vec{n} \equiv (n_1, \cdots, n_H)$, there are R instances of the label r are present in $a_{r\cdots r n_1 \cdots n_H}$ (e.g. $a_{(2|n_1 n_2)} = a_{r r n_1 n_2}$), and $a_{r\cdots r n_1\cdots n_H}$ is fully symmetric in the subscript (e.g. $a_{nnr} = a_{nrn} = a_{rnn}$). Using this, we may now write

$$X_A^{(\text{eff})} = \left[\frac{\kappa}{\sqrt{\pi r_c}}\right]^{H+R-2} \sum_{n_1,\cdots,n_H=0}^{+\infty} a_{(R|n_1\cdots n_H)} \left(\partial_\mu^2,\ \hat{h}^{(n_1)} \cdots \hat{h}^{(n_H)},\ \left[\hat{r}^{(0)}\right]^R\right) \ . \tag{4.57}$$

To simplify this expression further, we define a KK decomposition operator $\mathcal{X}_{(\vec{n})}[\bullet]$. The KK decomposition operator maps a product of $\hat{h}_{\mu\nu}$ and $\hat{r}$ fields to an analogous product of 4D spin-2 fields $\hat{h}_{\mu\nu}^{(n_i)}$ labeled by KK numbers $\vec{n} = (n_1, \cdots, n_H)$ and 4D radion fields $\hat{r}^{(0)}$. More specifically, $\mathcal{X}$ maps all $\hat{r}$ in its argument to $\hat{r}^{(0)}$ and applies the specified KK labels to the $\hat{h}_{\mu\nu}$ fields ($\hat{h}_{\mu\nu} \mapsto \hat{h}_{\mu\nu}^{(n_i)}$) per term according to the following prescription: the labels are applied left to right in the order that they occur in $\vec{n}$, and are applied to $\hat{h}_{\mu\nu}$ fields of the form $(\partial_y \hat{h})$ before being applied to all other $\hat{h}_{\mu\nu}$ fields. (This prescription ensures we correctly keep track of KK number relative to the soon-to-be-defined quantity b.) After KK number assignment, any 4D derivatives ∂_μ in the argument of $\mathcal{X}$ are kept as is, while each extra-dimensional derivative ∂_y is replaced by $1/r_c$.

Using $\mathcal{X}$, we rewrite the A-type expression:

$$X_A^{(\text{eff})} = \left[\frac{\kappa}{\sqrt{\pi r_c}}\right]^{H+R-2} \sum_{n_1,\cdots,n_H=0}^{+\infty} a_{(R|n_1\cdots n_H)} \cdot \mathcal{X}_{(n_1\cdots n_H)}\left[\overline{X}_A\right] \ . \tag{4.58}$$

This completes the schematic A-type procedure. B-type terms admit a similar reorganization. First, we KK decompose and integrate X_B to obtain

$$\begin{aligned} X_B^{(\text{eff})} =& \frac{r_c}{(\pi r_c)^{(H+R)/2}} \kappa^{(H+R-2)} \sum_{n_1,\cdots,n_H=0}^{+\infty} \left(1,\ \hat{h}^{(n_1)} \cdots \hat{h}^{(n_H)},\ \left[\hat{r}^{(0)}\right]^R\right) \\ & \times e^{-R\pi k r_c} \int d\varphi\ \varepsilon^{2(R-2)} (\partial_\varphi \psi_{n_1})(\partial_\varphi \psi_{n_2}) \psi_{n_3} \cdots \psi_{n_H} \left[\psi_0\right]^R \ . \end{aligned} \tag{4.59}$$

We summarize the extra-dimensional overlap integral as a unitless quantity b:

$$\begin{aligned} b_{(R|\vec{n})} &\equiv b_{r\cdots r n_1' n_2' n_3 \cdots n_H} \ , \\ &\equiv \frac{1}{\pi} e^{-R\pi k r_c} \int_{-\pi}^{+\pi} d\varphi\ \varepsilon^{2(R-2)} (\partial_\varphi \psi_{n_1})(\partial_\varphi \psi_{n_2}) \psi_{n_3} \cdots \psi_{n_H} \left[\psi_0\right]^R \ , \end{aligned} \tag{4.60}$$

where primes on a KK index in the subscript of $b_{r\cdots r n_1' n_2' n_3 \cdots n_H}$ indicates differentiation of the corresponding wavefunction and $b_{r\cdots r n_1' n_2' n_3 \cdots n_H}$ is symmetric in its subscript (e.g. $b_{rn'n'n} = b_{nn'rn'}$ and so-on). When expressed in $b_{(R|\vec{n})}$ form, the first two indices listed in $\vec{n}$ will be primed in $b_{r\cdots r n_1' n_2' n_3 \cdots n_H}$ form. With this definition,

$$X_B^{(\text{eff})} = \left[\frac{\kappa}{\sqrt{\pi r_c}}\right]^{H+R-2} \sum_{n_1,\cdots,n_H=0}^{+\infty} b_{(R|n_1 n_2 n_3 \cdots n_H)} \frac{1}{r_c^2} \left(1,\ \hat{h}^{(n_1)} \cdots \hat{h}^{(n_H)},\ \left[\hat{r}^{(0)}\right]^R\right) \ , \tag{4.61}$$

and, via the KK decomposition operator $\mathcal{X}$,

$$X_B^{(\text{eff})} = \left[\frac{\kappa}{\sqrt{\pi r_c}}\right]^{H+R-2} \sum_{n_1,\cdots,n_H=0}^{+\infty} b_{(R|n_1 n_2 n_3 \cdots n_H)} \cdot \mathcal{X}_{(n_1\cdots n_H)}\left[\overline{X}_B\right] \ , \tag{4.62}$$

where we recall that $\mathcal{X}$ maps ∂_y to $1/r_c$ after KK number assignment. This completes the schematic B-type procedure.

We now connect these procedures to the 4D effective RS1 Lagrangian $\mathcal{L}_{4D}^{(\text{RS,eff})}$, following the arrangement of the 5D Lagrangian described in Sec. 3.4. Suppose we collect all terms from the WFE RS1 Lagrangian $\mathcal{L}_{5D}^{(\text{RS})}$ that contain H $\hat{h}_{\mu\nu}$ fields and R $\hat{r}$ fields. Label this collection $\mathcal{L}^{(\text{RS})}_{hH_rR}$. In general, we can subdivide those terms into two sets based on their derivative content, i.e. whether they are A-type or B-type.

$$\mathcal{L}^{(\text{RS})}_{hH_rR} = \mathcal{L}^{(\text{RS})}_{A:hH_rR} + \mathcal{L}^{(\text{RS})}_{B:hH_rR} \,. \tag{4.63}$$

We may go a step further by using our existing knowledge to preemptively extract powers of the expansion parameter κ and any exponential coefficients:

$$\mathcal{L}^{(\text{RS})}_{hH_rR} = \kappa^{(H+R-2)}\left[e^{-R\pi kr_c}\varepsilon^{2(R-1)}\,\overline{\mathcal{L}}_{A:hH_rR} + e^{-R\pi kr_c}\varepsilon^{2(R-2)}\,\overline{\mathcal{L}}_{B:hH_rR}\right] . \tag{4.64}$$

Finally, we can apply the schematic procedures described above to obtain a succinct expression for the effective Lagrangian with H $\hat{h}_{\mu\nu}$ fields and R $\hat{r}$ fields:

$$\boxed{\mathcal{L}^{(\text{RS,eff})}_{hH_rR} = \left[\frac{\kappa}{\sqrt{\pi r_c}}\right]^{(H+R-2)} \sum_{\vec{n}=\vec{0}}^{+\infty}\left\{a_{(R|\vec{n})}\cdot\mathcal{X}_{(\vec{n})}\Big[\,\overline{\mathcal{L}}_{A:hH_rR}\Big] + b_{(R|\vec{n})}\cdot\mathcal{X}_{(\vec{n})}\Big[\,\overline{\mathcal{L}}_{B:hH_rR}\Big]\right\}} \,. \tag{4.65}$$

Computationally, a key feature of this Lagrangian is how the dependence on the physical variables arrange themselves. Consider the set $\{M_{\text{Pl}}, kr_c, m_1\}$. The parameter kr_c determines the wavefunctions $\{\psi_n\}$ and spectrum $\{\mu_n\}\equiv\{m_n r_c\}$, and thus $\{a_{(R|\vec{n})}, b_{(R|\vec{n})}\}$ as well. Additionally fixing the value of m_1 determines $r_c = \mu_1/m_1$ and $k = (kr_c)m_1/\mu_1$. Finally, fixing M_{Pl} determines the prefactor $\kappa/\sqrt{\pi r_c} = \kappa_{4\text{D}}/\psi_0 = 2/(M_{\text{Pl}}\psi_0)$. Therefore, referring back to the specific form of Eq. (4.65), once kr_c is fixed, changing m_1 only affects the relative importance of A-type vs. B-type terms via factors of r_c introduced by $\mathcal{X}_{(\vec{n})}[\bullet]$ and changing M_{Pl} only affects the interaction's overall strength via $[\kappa/\sqrt{\pi r_c}]^{(H+R-2)}$. Alternatively, by fixing κ and r_c instead, the couplings $\{a_{(R|\vec{n})}, b_{(R|\vec{n})}\}$ encapsulate the effect of varying k.

While the $a_{(R|\vec{n})}$ and $b_{(R|\vec{n})}$ forms are useful when deriving Eq. (4.65), the alternate notations introduced in Eqs. (4.56) and (4.60) are more useful in practice. They are special instances of a more general structure x, which we define as:

$$x^{(p)}_{r\cdots m'\cdots n} \equiv \frac{1}{\pi}\int_{-\pi}^{+\pi} d\varphi \quad \varepsilon^p(\partial_\varphi\psi_m)\cdots\psi_n\left[e^{-\pi kr_c}\varepsilon^{+2}\psi_0\right]^R \tag{4.66}$$

to which we add an additional factor of $(\partial_\varphi|\varphi|)$ if there is an odd number of primed labels (without this factor, the quantity would automatically vanish). In terms of x, the A-type and B-type couplings are special cases equal to

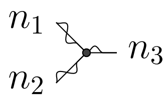
$$a_{{}_rR\,m'\cdots n} \equiv x^{(-2)}_{{}_rR_{m'\cdots n}} \qquad\qquad b_{{}_rR\,m'\cdots n} \equiv x^{(-4)}_{{}_rR_{m'\cdots n}} \tag{4.67}$$

where this generalization now allows A-type and B-type couplings to contain various numbers of differentiated wavefunctions in principle. This dissertation concerns tree-level massive spin-2 scattering, which is calculated from diagrams of the forms described in Section 2.5. Consequently, the relevant couplings are cubic and quartic couplings of the forms

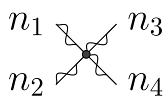
$$a_{lmn} \qquad b_{l'm'n} \qquad b_{m'n'r} \qquad a_{klmn} \qquad b_{k'l'mn} \tag{4.68}$$

Note that an analogous cubic A-type radion coupling does not occur in the RS1 model.

Pictorially, we indicate the vertices associated with these couplings as small filled circles attached to the appropriate number of particle lines, e.g. the relevant spin-2 exclusive couplings arise from



$$\supset \qquad a_{n_1n_2n_3} \qquad b_{n'_{\mathcal{P}[1]}n'_{\mathcal{P}[2]}n_{\mathcal{P}[3]}} \tag{4.69}$$



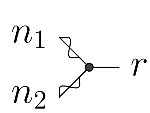
$$\supset \qquad a_{n_1n_2n_3n_4} \qquad b_{n'_{\mathcal{P}[1]}n'_{\mathcal{P}[2]}n_{\mathcal{P}[3]}n_{\mathcal{P}[4]}} \tag{4.70}$$

where overlapping straight and wavy lines indicate a spin-2 particle, and $\mathcal{P}$ is a generic permutation of the indices. If we set $n_3 = 0$ in the triple spin-2 coupling, the corresponding wavefunction ψ_0 is flat; either ψ_0 is differentiated (in which case the integral vanishes) or it can be factored out of the y-integral thereby allowing us to invoke the wavefunction orthogonality relations on the remaining wavefunction pair. In this way, the triple spin-2 couplings imply that the massless 4D graviton couples diagonally to the other spin-2 states, as required by 4D general covariance:

$$\begin{aligned} a_{n_1n_20} &= \psi_0\, \delta_{n_1,n_2}\ , \\ b_{n'_1n'_20} &= \mu_{n_1}^2 \psi_0\, \delta_{n_1,n_2}\ , \\ b_{0'n'_1n_2} &= 0\ . \end{aligned} \tag{4.71}$$

The Sturm-Liouville problem Eq. (4.36) that defines the wavefunctions $\{\psi_n\}$ also relates various A-type and B-type couplings to each other, which we be explored further in the next section.

When calculating matrix elements of massive KK mode scattering, we must also consider radion-mediated diagrams. As mentioned previously, the RS1 model lacks a cubic A-type (KK mode)-(KK mode)-radion coupling. Furthermore, note that the additional ε^{+2} exponential factor in the integrand of $b_{n'_1n'_2r}$ due to the radion field (as in Eq. (4.66)) prevents the use of the orthonormality relations Eqs. (4.37) and (4.38); therefore, the radion typically couples non-diagonally to massive spin-2 modes. Pictorially,



$$\supset \quad b_{n'_1n'_2r} \tag{4.72}$$

where unadorned straight lines indicate a radion.

4.3.3 Summary of Results

Section 3.4 summarized all terms in the WFE RS1 Lagrangian $\mathcal{L}_{\text{5D}}^{(\text{RS})}$ that contain four or fewer fields. In particular, it listed explicit expressions for all relevant $\overline{\mathcal{L}}_A$ and $\overline{\mathcal{L}}_B$. Application of Eq. (4.65) to all of these combinations yields a WFE 4D effective Lagrangian of the following form:

$$\mathcal{L}_{4D}^{(\text{RS,eff})} = \mathcal{L}_{hh}^{(\text{eff})} + \mathcal{L}_{rr}^{(\text{eff})} + \mathcal{L}_{hhh}^{(\text{eff})} + \cdots + \mathcal{L}_{rrr}^{(\text{eff})} + \mathcal{L}_{hhhh}^{(\text{eff})} + \cdots + \mathcal{L}_{rrrr}^{(\text{eff})} + \mathcal{O}(\kappa^3)\ . \tag{4.73}$$

Explicitly, we find, at quadratic order,

$$\begin{aligned}\mathcal{L}_{hh}^{(\text{eff})} =& \sum_{n=0}^{+\infty} \left[-\hat{h}_{\mu\nu}^{(n)} (\partial^\mu \partial^\nu \hat{h}^{(n)}) + \hat{h}_{\mu\nu}^{(n)} (\partial^\mu \partial_\rho \hat{h}^{(n)\rho\nu}) - \frac{1}{2} \hat{h}_{\mu\nu}^{(n)} (\Box \hat{h}^{(n)\mu\nu}) + \frac{1}{2} \hat{h}^{(n)} (\Box \hat{h}^{(n)}) \right] \\ & + m_n^2 \left[-\frac{1}{2} [\![\hat{h}^{(n)} \hat{h}^{(n)}]\!] + \frac{1}{2} [\![\hat{h}^{(n)}]\!] [\![\hat{h}^{(n)}]\!] \right]\ ,\end{aligned} \tag{4.74}$$

$$\mathcal{L}_{rr}^{(\text{eff})} = \frac{1}{2} (\partial_\mu \hat{r}^{(0)}) (\partial^\mu \hat{r}^{(0)})\ , \tag{4.75}$$

and, at cubic order,

$$\mathcal{L}_{hhh}^{(\text{eff})} = \frac{\kappa}{\sqrt{\pi r_c}} \sum_{l,m,n=0}^{+\infty} \left\{ a_{(0|lmn)} \cdot \mathcal{X}_{(lmn)} \big[\overline{\mathcal{L}}_{A:hhh} \big] + b_{(0|lmn)} \cdot \mathcal{X}_{(lmn)} \big[\overline{\mathcal{L}}_{B:hhh} \big] \right\}\ , \tag{4.76}$$

$$\mathcal{L}_{hhr}^{(\text{eff})} = \frac{\kappa}{\sqrt{\pi r_c}} \sum_{m,n=0}^{+\infty} \left\{ b_{(1|mn)} \cdot \mathcal{X}_{(mn)} \big[\overline{\mathcal{L}}_{B:hhr} \big] \right\}\ , \tag{4.77}$$

$$\mathcal{L}_{hrr}^{(\text{eff})} = \frac{\kappa}{\sqrt{\pi r_c}} \sum_{n=0}^{+\infty} \left\{ a_{(2|n)} \cdot \mathcal{X}_{(n)} \big[\overline{\mathcal{L}}_{A:hrr} \big] \right\}\ , \tag{4.78}$$

$$\mathcal{L}_{rrr}^{(\text{eff})} = \frac{\kappa}{\sqrt{\pi r_c}} \left\{ a_{(3)} \cdot \mathcal{X} \big[\overline{\mathcal{L}}_{A:rrr} \big] \right\}\ , \tag{4.79}$$

and, at quartic order,

$$\mathcal{L}^{(\text{eff})}_{hhhh} = \left[\frac{\kappa}{\sqrt{\pi r_c}}\right]^2 \sum_{k,l,m,n=0}^{+\infty} \left\{ a_{(klmn)} \cdot \mathcal{X}_{(klmn)}\big[\overline{\mathcal{L}}_{A:hhhh}\big] + b_{(klmn)} \cdot \mathcal{X}_{(klmn)}\big[\overline{\mathcal{L}}_{B:hhhh}\big] \right\}, \tag{4.80}$$

$$\mathcal{L}^{(\text{eff})}_{hhhr} = \left[\frac{\kappa}{\sqrt{\pi r_c}}\right]^2 \sum_{l,m,n=0}^{+\infty} \left\{ b_{(1|lmn)} \cdot \mathcal{X}_{(lmn)}\big[\overline{\mathcal{L}}_{B:hhhr}\big] \right\}, \tag{4.81}$$

$$\mathcal{L}^{(\text{eff})}_{hhrr} = \left[\frac{\kappa}{\sqrt{\pi r_c}}\right]^2 \sum_{m,n=0}^{+\infty} \left\{ a_{(2|mn)} \cdot \mathcal{X}_{(mn)}\big[\overline{\mathcal{L}}_{A:hhrr}\big] + b_{(2|mn)} \cdot \mathcal{X}_{(mn)}\big[\overline{\mathcal{L}}_{B:hhrr}\big] \right\}, \tag{4.82}$$

$$\mathcal{L}^{(\text{eff})}_{hrrr} = \left[\frac{\kappa}{\sqrt{\pi r_c}}\right]^2 \sum_{n=0}^{+\infty} \left\{ a_{(3|n)} \cdot \mathcal{X}_{(n)}\big[\overline{\mathcal{L}}_{A:hrrr}\big] \right\}, \tag{4.83}$$

$$\mathcal{L}^{(\text{eff})}_{rrrr} = \left[\frac{\kappa}{\sqrt{\pi r_c}}\right]^2 \left\{ a_{(4)} \cdot \mathcal{X}\big[\overline{\mathcal{L}}_{A:rrrr}\big] \right\}. \tag{4.84}$$

The quantity $a_{(R|\vec{n})}$ is defined in Eq. (4.56), $b_{(R|\vec{n})}$ is defined in Eq. (4.60), and the KK decomposition operator $\mathcal{X}$ is introduced below Eq. (4.57).

4.3.4 Interaction Vertices

The 4D effective interaction Lagrangians $\mathcal{L}^{(\text{eff})}_{h^H r^R}$ of the previous subsection imply interaction vertices $v_{h^H r^R}$. When deriving those vertices, we apply functional derivatives to the interaction Lagrangians, which should in principle be performed according to the definitions

$$\frac{\delta}{\delta \hat{r}^{(0)}}\left[\hat{r}^{(0)}\right] = 1 \qquad\qquad \frac{\delta}{\delta \hat{h}^{(n_1)}_{\alpha_1\beta_1}}\left[\hat{h}^{(n)}_{\alpha\beta}\right] = \frac{1}{2}\left(\eta^{\alpha_1}_{\alpha}\eta^{\beta_1}_{\beta} + \eta^{\alpha_1}_{\beta}\eta^{\beta_1}_{\alpha}\right)\delta_{n,n_1} \tag{4.85}$$

However, in practice each pair of spin-2 Lorentz indices in these vertices will end up projected onto either a polarization tensor or a propagator, all of which have already had their Lorentz indices symmetrized. Therefore, we need not additionally symmetrize the indices in Eq. (4.85) and in doing so can avoid introducing terms that will otherwise complicate algebraic manipulations. That is, effectively,

$$\frac{\delta}{\delta \hat{h}^{(n_1)}_{\alpha_1\beta_1}}\left[\hat{h}^{(n)}_{\alpha\beta}\right] \overset{\text{in practice}}{=} \eta^{\alpha_1}_{\alpha}\eta^{\beta_1}_{\beta}\delta_{n,n_1} \tag{4.86}$$

Furthermore, each 4D derivative ∂_μ acting on the field being differentiated is replaced by $-i\alpha p_\mu$, where $\alpha = \pm 1$ if the corresponding 4-momentum is entering (leaving) the vertex. In order to keep track of which 4-momenta are associated with which fields, we introduce labels on the functional derivative fields. For the spin-2 fields $\hat{h}^{(n)}_{\mu\nu}$, this can be accomplished via the subscripts we already utilized in Eq. (4.85). For the radion fields $\hat{r}^{(0)}$, we add an additional subscript, e.g. $\hat{r}_1^{(0)}$. As long as the subscripts are chosen so that they uniquely label fields connected to a given vertex, all is well.

The conversion of a typical term of the 4D effective Lagrangian into the corresponding interaction vertex proceeds like so:

$$v_{hrr} \supset i\frac{\delta}{\delta\hat{r}_2^{(0)}}\frac{\delta}{\delta\hat{r}_1^{(0)}}\frac{\delta}{\delta\hat{h}_{\alpha_3\beta_3}^{(n_3)}}\left[\frac{\kappa}{\sqrt{\pi r_c}}\sum_{n=0}^{+\infty} a_{nrr}\,\hat{h}_{\mu\nu}^{(n)}(\partial^\mu\hat{r}^{(0)})(\partial^\nu\hat{r}^{(0)})\right] \tag{4.87}$$

$$= i\frac{\kappa}{\sqrt{\pi r_c}}\,a_{n_3rr}\,\frac{\delta}{\delta\hat{r}_2^{(0)}}\frac{\delta}{\delta\hat{r}_1^{(0)}}\left[\eta_\mu^{\alpha_3}\eta_\nu^{\beta_3}(\partial^\mu\hat{r}^{(0)})(\partial^\nu\hat{r}^{(0)})\right] \tag{4.88}$$

$$= i\frac{\kappa}{\sqrt{\pi r_c}}\,a_{n_3rr}\,\frac{\delta}{\delta\hat{r}_2^{(0)}}\left[\eta_\mu^{\alpha_3}\eta_\nu^{\beta_3}(-i\alpha_2 p_2^\mu)(\partial^\nu\hat{r}^{(0)}) + \eta_\mu^{\alpha_3}\eta_\nu^{\beta_3}(\partial^\mu\hat{r}^{(0)})(-i\alpha_2 p_2^\nu)\right] \tag{4.89}$$

$$= i\frac{\kappa}{\sqrt{\pi r_c}}\,a_{n_3rr}\left[\eta_\mu^{\alpha_3}\eta_\nu^{\beta_3}(-i\alpha_2 p_2^\mu)(-i\alpha_1 p_1^\nu) + \eta_\mu^{\alpha_3}\eta_\nu^{\beta_3}(-i\alpha_1 p_1^\mu)(-i\alpha_2 p_2^\nu)\right] \tag{4.90}$$

$$= -i\frac{\kappa}{\sqrt{\pi r_c}}\,a_{n_3rr}\,\alpha_1\alpha_2(p_1^{\alpha_3}p_2^{\beta_3} + p_1^{\beta_3}p_2^{\alpha_3}) \tag{4.91}$$

where 1 and 2 label attached radion lines and 3 labels an attached n_3th spin-2 KK mode.

4.3.5 The Large kr_c Limit[7]

Consider how the aforementioned wavefunctions and couplings behave in the limit that kr_c is large. In this limit, the behavior of the irregular Bessel functions Y_ν causes the coefficients $b_{n\nu}$ in Eq. (4.33) to be small, such that the wavefunctions of Eq. (4.32) (having nonzero KK mode number n) can be approximated as

$$\psi_n(\varphi) \approx \frac{1}{N_n}e^{+2kr_c|\phi|}J_2\left[x_n e^{kr_c(|\phi|-\pi)}\right]\ , \tag{4.92}$$

where x_n is the nth root of J_1 and

$$N_n \approx \frac{e^{\pi kr_c}}{\sqrt{\pi kr_c}}J_0(x_n)\ , \tag{4.93}$$

corresponding to a state with mass

$$m_n \approx x_n k\, e^{-\pi kr_c}\ . \tag{4.94}$$

This limit—called the "large kr_c limit"—is a good approximation when $kr_c \gtrsim 3$ and is popular in the literature. The above expressions can be further simplified by replacing φ with the quantity $u_n \equiv x_n e^{kr_c(\varphi-\pi)}$. In terms of u_n, the $n \neq 0$ wavefunction factorizes into separate u_n and kr_c-dependent pieces,

$$\psi_n(u_n) \approx \frac{\sqrt{\pi}}{x_n^2\,|J_0(x_n)|}\left[u_n^2\,J_2(u_n)\right]\cdot\sqrt{kr_c}\,e^{\pi kr_c}\ . \tag{4.95}$$

[7] This subsection was originally published as Appendices F.1-2 of [18].

This factorization is not unique to ψ_n: for generic $j \neq 0$,

$$\psi_j(u_n) \approx \frac{\sqrt{\pi}}{x_n^2\,|J_0(x_j)|}\,\left[u_n^2\,J_2\left(\frac{x_j}{x_n}u_n\right)\right]\cdot\sqrt{kr_c}\,e^{\pi k r_c}\;, \tag{4.96}$$

and,

$$(\partial_\varphi\psi_j)(u_n) \approx \frac{\sqrt{\pi}\,x_j}{x_n^3\,|J_0(x_j)|}\,\left[u^3\,J_1\left(\frac{x_j}{x_n}u_n\right)\right]\,(kr_c)^{3/2}\,e^{\pi k r_c}\;. \tag{4.97}$$

Meanwhile, the large krc approximation of the zero mode wavefunction is

$$\psi_0 \sim \sqrt{\pi k r_c}\;. \tag{4.98}$$

We can also rewrite the coupling integrals as integrals over u_n instead of φ and (in doing so) factor any kr_c-dependence from the integral. Specifically, we convert φ integrals of the form

$$\int_{-\pi}^{+\pi} d\varphi\;e^{-Akrc|\varphi|}\,f(|\varphi|) = 2\int_0^{+\pi} d\varphi\;e^{-Akrc|\varphi|}\,f(\varphi)\;, \tag{4.99}$$

to $u_n \equiv x_n e^{krc(\varphi-\pi)}$ integrals, noting that $d\varphi = du_n/(kr_c u_n)$,

$$\int_{-\pi}^{+\pi} d\varphi\;e^{-Akrc|\varphi|}\,f(|\varphi|) = \frac{2x_n^A e^{-Akr_c\pi}}{kr_c}\cdot\left[\int_{u_n(0)}^{u_n(\pi)}\frac{du_n}{u_n^{A+1}}\;f\left(\varphi(u_n)\right)\right]\;, \tag{4.100}$$

for any n. In the large kr_c limit, the integration limits become independent of kr_c,

$$u_n(0) = e^{-kr_c\pi}x_n \to 0 \qquad\qquad u_n(\pi) = x_n\;. \tag{4.101}$$

and thus the integral over u_n does not depend on kr_c. By combining all of the preceding elements, we can factor all kr_c-dependence out of the coupling integrals in the large kr_c limit, and we find

$$a_{nnnn} \approx C_{nnnn}\,(kr_c)\,e^{2\pi k r_c}\;, \tag{4.102}$$

$$a_{nn0} \approx C_{nn0}\,\sqrt{kr_c}\;, \tag{4.103}$$

$$b_{n'n'r} \approx C_{nnr}\,(kr_c)^{5/2}\,e^{-\pi k r_c}\;, \tag{4.104}$$

$$a_{nnj} \approx C_{nnj}\,\sqrt{kr_c}\,e^{\pi k r_c}\;, \tag{4.105}$$

where the coefficients C are given by the following kr_c-independent integrals:

$$C_{nnnn} \equiv \left[\frac{2\pi}{x_n^6\,J_0(x_n)^4}\int_0^{x_n} du_n\;u_n^5\,J_2(u_n)^4\right]\;, \tag{4.106}$$

$$C_{nn0} \equiv \left[\frac{2\sqrt{\pi}}{x_n^2\,J_0(x_n)^2}\int_0^{x_n} du_n\;u_n\,J_2(u_n)^2\right]\;, \tag{4.107}$$

$$C_{nnr} \equiv \left[\frac{2\sqrt{\pi}}{x_n^2\,J_0(x_n)^2}\int_0^{x_n} du_n\;u_n^3\,J_1(u_n)^2\right]\;, \tag{4.108}$$

$$C_{nnj} \equiv \left[\frac{2\sqrt{\pi}}{x_n^4\,|J_0(x_j)|\,J_0(x_n)^2}\int_0^{x_n} du_n\;u_n^3\,J_2(u_n)^2\,J_2\left(\frac{x_j}{x_n}u_n\right)\right] \tag{4.109}$$

Although we utilize exact expressions when investigating the high-energy behavior of matrix elements, the approximate expressions derived in this subsection will be useful when we calculate the strong-coupling scale of the RS1 model in the next chapter.

4.4 Sum Rules Between Couplings and Masses[8]

This section derives relationships between the spin-2 exclusive couplings and spin-2 KK spectrum $\{\mu_n\}$ that are relevant to tree-level 2-to-2 massive KK mode scattering. We briefly consider the implications of completeness before deriving a means of expressing all cubic and quartic (spin-2 exclusive) B-type couplings in terms of A-type couplings. These B-to-A formulas reduce the problem of finding amplitude-relevant formulas to the problem of simplifying sums of the form $\sum_j \mu_j^{2i} a_{klj} a_{mnj}$. The relevant sum rules are derived and then summarized in the final two subsections for the inelastic and elastic cases.

4.4.1 Applications of Completeness

The completeness relation Eq. (4.39) allows us to collapse certain sums of cubic coupling products into a single quartic coupling. For example, a pair of cubic A-type couplings can be combined into a quartic A-type coupling:

$$\sum_j a_{jkl} a_{jmn} = \sum_j \left[\frac{1}{\pi}\int d\varphi_1 \;\varepsilon(\varphi_1)^{-2}\,\psi_j(\varphi_1)\,\psi_k(\varphi_1)\,\psi_l(\varphi_1)\right] \times \left[\frac{1}{\pi}\int d\varphi_2 \;\varepsilon(\varphi_2)^{-2}\,\psi_j(\varphi_2)\,\psi_k(\varphi_2)\,\psi_l(\varphi_2)\right] \tag{4.110}$$

$$= \frac{1}{\pi^2}\int d\varphi_1\, d\varphi_2 \;\varepsilon(\varphi_1)^{-2}\,\varepsilon(\varphi_2)^{-2}\,\psi_k(\varphi_1)\,\psi_l(\varphi_1)\,\psi_m(\varphi_2)\,\psi_n(\varphi_2)\left[\sum_j \psi_j(\varphi_1)\,\psi_j(\varphi_2)\right] \tag{4.111}$$

$$= \frac{1}{\pi^2}\int d\varphi_1\, d\varphi_2 \;\varepsilon(\varphi_1)^{-2}\,\varepsilon(\varphi_2)^{-2}\,\psi_k(\varphi_1)\,\psi_l(\varphi_1)\,\psi_m(\varphi_2)\,\psi_n(\varphi_2)\left[\pi\varepsilon(\varphi_2)^{+2}\,\delta(\varphi_2-\varphi_1)\right] \tag{4.112}$$

$$= \frac{1}{\pi}\int d\varphi_1 \;\varepsilon(\varphi_1)^{-2}\,\psi_k(\varphi_1)\,\psi_l(\varphi_1)\,\psi_m(\varphi_1)\,\psi_n(\varphi_1) = a_{klmn} \tag{4.113}$$

By applying this same procedure to other A-type and B-type couplings, we find

$$a_{klmn} = \sum_j a_{jkl} a_{jmn} = \sum_j a_{jkm} a_{jln} = \sum_j a_{jkn} a_{jlm} \tag{4.114}$$

$$b_{k'l'mn} = \sum_j b_{k'l'j} a_{jmn} \tag{4.115}$$

Furthermore, by combining cubic B-type couplings in this same way, we define an important new integral that will be present in many of our derivations:

$$c_{klmn} \equiv x^{(-6)}_{k'l'm'n'} = \frac{1}{\pi}\int d\varphi \;\varepsilon^{-6}(\partial_\varphi\psi_k)(\partial_\varphi\psi_l)(\partial_\varphi\psi_m)(\partial_\varphi\psi_n) \tag{4.116}$$

$$= \sum_j b_{k'l'j} b_{jm'n'} = \sum_j b_{k'm'j} b_{jl'n'} = \sum_j b_{k'n'j} b_{jl'm'}$$

[8] The material of this section is entirely new to this dissertation. It generalizes results first published in [17] and later generalized (but not to the same extent) in [18].

where the generic coupling integral x is defined in Eq. 4.66. Another object that will be useful throughout the rest of the chapter is the symbol $\mathcal{D} \equiv \varepsilon^{-4}\partial_\varphi$, which is a combination of quantities that is often present as a result of the Sturm-Liouville equation.

4.4.2 B-to-A Formulas

This subsection details how to eliminate all B-type couplings in favor of A-type couplings. To begin, we note we can absorb a factor of μ^2 into A-type couplings with help from the Sturm-Liouville equation. A standard application of this technique proceeds as follows:

$$\begin{aligned}
\mu_n^2 a_{lmn} &= \frac{1}{\pi}\int d\varphi\ \varepsilon^{-2}\psi_l\psi_m\left[\mu_n^2\psi_n\right] && (4.117)\\
&= \frac{1}{\pi}\int d\varphi\ \varepsilon^{-2}\psi_l\psi_m\Big[-\varepsilon^{+2}\partial_\varphi(\mathcal{D}\psi_n)\Big] && (4.118)\\
&= \frac{1}{\pi}\int d\varphi\ \partial_\varphi\left[\psi_l\psi_m\right](\mathcal{D}\psi_n) && (4.119)\\
&= \frac{1}{\pi}\int d\varphi\ \varepsilon^{-4}(\partial_\varphi\psi_l)\psi_m(\partial_\varphi\psi_n) + \frac{1}{\pi}\int d\varphi\ \varepsilon^{-4}\psi_l(\partial_\varphi\psi_m)(\partial_\varphi\psi_n) && (4.120)\\
&= b_{l'mn'} + b_{lm'n'} && (4.121)
\end{aligned}$$

where integration by parts was utilized between Eqs. (4.118) and (4.119). This and the equivalent calculation with the quartic A-type coupling yield

$$\begin{aligned}
\mu_n^2 a_{lmn} &= b_{l'mn'} + b_{lm'n'} && (4.122)\\
\mu_n^2 a_{klmn} &= b_{k'lmn'} + b_{kl'mn'} + b_{klm'n'} && (4.123)
\end{aligned}$$

By considering different permutations of KK indices, each of these equations corresponds to three and four unique constraints respectively. Because there are only three unique cubic B-type couplings with KK indices l, m, and n (specifically, $b_{l'm'n}$, $b_{l'mn'}$, and $b_{lm'n'}$), Eq. (4.122) can be inverted to yield

$$b_{l'm'n} = \frac{1}{2}\left[\mu_l^2 + \mu_m^2 - \mu_n^2\right] a_{lmn} \tag{4.124}$$

with which we can eliminate all cubic B-type couplings in favor of the cubic A-type coupling.

Because there are six unique B-type couplings with KK indices k, l, m, and n, we require additional constraints before we can rewrite all quartic B-type couplings in terms of the quartic A-type coupling. Using the cubic coupling equation Eq. (4.122) with completeness yields,

$$\begin{aligned}
b_{k'l'mn} &= \sum_j b_{k'l'j}\, a_{jmn} && (4.125)\\
&= \frac{1}{2}\sum_j \left[\mu_k^2 + \mu_l^2 - \mu_j^2\right] a_{jkl}\, a_{jmn} && (4.126)\\
&= \frac{1}{2}\left[\mu_k^2 + \mu_l^2\right] a_{klmn} - \frac{1}{2}\sum_j \mu_j^2 a_{jkl}\, a_{jmn} && (4.127)
\end{aligned}$$

Similar expressions hold for $b_{k'lmn'}$, $b_{kl'mn'}$, and $b_{klm'n'}$, which when summed as in the RHS of the quartic coupling equation Eq. (4.123) then imply

$$\mu_n^2 a_{klmn} = \frac{1}{2}\left[\mu_k^2 + \mu_l^2 + \mu_m^2 + 3\mu_n^2\right] a_{klmn} - \frac{3}{2}\sum_j \mu_j^2 a_{jkl} a_{jmn} \tag{4.128}$$

such that, by solving for the undetermined sum,

$$\sum_j \mu_j^2 a_{jkl} a_{jmn} = \frac{1}{3}\left[\mu_k^2 + \mu_l^2 + \mu_m^2 + \mu_n^2\right] a_{klmn} \equiv \frac{1}{3}\vec{\mu}^{\,2} a_{klmn} \tag{4.129}$$

where $\vec{\mu}^{\,2} \equiv \mu_k^2 + \mu_l^2 + \mu_m^2 + \mu_n^2$. Note that the RHS is symmetric in all indices despite the LHS not obviously exibiting such a symmetry. Utilizing Eq. (4.129) in Eq. (4.127) then allows us to finally rewrite all quartic B-type couplings in terms of A-type couplings:

$$b_{k'l'mn} = \frac{1}{6}\left[2\mu_k^2 + 2\mu_l^2 - \mu_m^2 - \mu_n^2\right] a_{klmn} \tag{4.130}$$

This and Eq. (4.124) comprise the desired B-to-A formulas.

The B-to-A formulas greatly reduce the number of relations we must consider. For example, when calculating a tree-level 2-to-2 KK mode scattering amplitude, we encounter quantities such as

$$\sum_j b_{k'l'j} a_{jmn} \qquad \sum_j b_{k'l'j} b_{m'n'j} \qquad \sum_j \mu_j^2 b_{k'l'j} b_{m'n'j} \tag{4.131}$$

where the indices $\{k, l, m, n\}$ are associated with external KK modes and the index j labels an intermediate KK mode that must be summed over in the course of summing over all diagrams. However, by converting all B-type couplings to A-type couplings, the quantities can be evaluated so long as we know instead how to evaluate

$$\sum_j a_{jkl} a_{jmn} \qquad \sum_j \mu_j^2 a_{jkl} a_{jmn} \qquad \sum_j \mu_j^4 a_{jkl} a_{jmn} \qquad \sum_j \mu_j^6 a_{jkl} a_{jmn} \tag{4.132}$$

Indeed, these are precisely the sums that are relevant to cancelling the high-energy growth of the KK mode scattering amplitudes, which is the goal of this dissertation. The remainder of this chapter is dedicated to rewriting these quantities in terms of the quartic A-type coupling and the integral c_{klmn} of Eq. (4.116). The first two of these rewrites were achieved in Eqs. (4.114) and (4.129) respectively. Therefore, we turn our focus to $\sum_j \mu_j^4 a_{jkl} a_{jmn}$ and then $\sum_j \mu_j^6 a_{jkl} a_{jmn}$.

4.4.3 The μ_j^4 Sum Rule

The $\sum_j \mu_j^4 a_{jkl} a_{jmn}$ relation is relatively straightforward. As defined in Eq. (4.116), we can rewrite c_{klmn} in terms of B-type cubic couplings, to which we can then apply the B-to-A formulas:

$$c_{klmn} = \sum_j b_{k'l'j} b_{m'n'j} \tag{4.133}$$

$$= \frac{1}{4} \sum_j \left[\mu_k^2 + \mu_l^2 - \mu_j^2\right] \left[\mu_m^2 + \mu_n^2 - \mu_j^2\right] a_{jkl} a_{jmn} \tag{4.134}$$

$$= \frac{1}{4}(\mu_k^2 + \mu_l^2)(\mu_m^2 + \mu_n^2) a_{klmn} - \frac{1}{4}(\vec{\mu}^{\,2}) \sum_j \mu_j^2 a_{jkl} a_{jmn} + \frac{1}{4} \sum_j \mu_j^4 a_{jkl} a_{jmn} \tag{4.135}$$

$$= \frac{1}{4} \left[(\mu_k^2 + \mu_l^2)(\mu_m^2 + \mu_n^2) - \frac{1}{3}(\vec{\mu}^{\,2})^2\right] a_{klmn} + \frac{1}{4} \sum_j \mu_j^4 a_{jkl} a_{jmn} \tag{4.136}$$

such that

$$\sum_j \mu_j^4 a_{jkl} a_{jmn} = 4 c_{klmn} + \left[\frac{1}{3}(\vec{\mu}^{\,2})^2 - (\mu_k^2 + \mu_l^2)(\mu_m^2 + \mu_n^2)\right] a_{klmn} \tag{4.137}$$

as desired. Deriving the $\sum_j \mu_j^6 a_{jkl} a_{jmn}$ relation requires significantly more work.

4.4.4 The μ_j^6 Sum Rule

As in the previous subsection, we begin our derivation by applying the B-to-A formulas to a sum of cubic B-type couplings:

$$\sum_j \mu_j^2 b_{jk'l'} b_{jm'n'} = \frac{1}{4} \sum_j \left[\mu_k^2 + \mu_l^2 - \mu_j^2\right] \left[\mu_m^2 + \mu_n^2 - \mu_j^2\right] \mu_j^2 a_{jkl} a_{jmn} \tag{4.138}$$

$$= \frac{1}{4} \sum_j \mu_j^6 a_{klj} a_{jmn} - \vec{\mu}^{\,2} c_{klmn} - \frac{1}{12}\Bigg[(\vec{\mu}^{\,2})^3 - 16 \sum_{x=k,l,m,n} \frac{\mu_k^2 \mu_l^2 \mu_m^2 \mu_n^2}{\mu_x^2} - 4\Big((\mu_k^2 - \mu_l^2)^2(\mu_m^2 + \mu_n^2) + (\mu_m^2 - \mu_n^2)^2(\mu_k^2 + \mu_l^2)\Big)\Bigg] a_{klmn} \tag{4.139}$$

Unlike the previous subsection, we do not yet have a simplification of the LHS of this expression. To obtain such a simplification, we would like to absorb the μ_j^2 factor into $b_{jm'n'}$, and thus we next consider:

$$\mu_l^2 b_{lm'n'} = \frac{1}{\pi}\int d\varphi\ \varepsilon^{-2}\left[\mu_l^2\varepsilon^{-2}\psi_l\right](\partial_\varphi\psi_m)(\partial_\varphi\psi_n) \tag{4.140}$$

$$= \frac{1}{\pi}\int d\varphi\ \varepsilon^{-4}(\partial_\varphi\psi_l)\,\partial_\varphi\left[\varepsilon^{-2}(\partial_\varphi\psi_m)(\partial_\varphi\psi_n)\right] \tag{4.141}$$

$$= \frac{1}{\pi}\int d\varphi\ \varepsilon^{-4}(\partial_\varphi\psi_l)\,\partial_\varphi\left[\varepsilon^{+6}(\mathcal{D}\psi_m)(\mathcal{D}\psi_n)\right] \tag{4.142}$$

$$= \frac{1}{\pi}\int d\varphi\ \varepsilon^{+2}(\partial_\varphi\psi_l)\left[+6(kr_c)(\partial_\varphi|\varphi|)(\mathcal{D}\psi_m)(\mathcal{D}\psi_n)\right.$$
$$\left.-\mu_m^2\varepsilon^{-2}\psi_m(\mathcal{D}\psi_n)-\mu_n^2\varepsilon^{-2}(\mathcal{D}\psi_m)\psi_n\right] \tag{4.143}$$

$$= \frac{6kr_c}{\pi}\int d\varphi\ (\partial_\varphi|\varphi|)\varepsilon^{+6}(\mathcal{D}\psi_l)(\mathcal{D}\psi_m)(\mathcal{D}\psi_n) - \mu_m^2 b_{l'mn'} - \mu_n^2 b_{l'm'n} \tag{4.144}$$

from which

$$\mu_l^2 b_{lm'n'} + \mu_m^2 b_{l'mn'} + \mu_n^2 b_{l'm'n} = 6(kr_c)x^{(-6)}_{l'm'n'} \tag{4.145}$$

After applying the B-to-A formulas, we can solve for the integral $x^{(-6)}_{l'm'n'}$ which we have not encountered previously:

$$6(kr_c)x^{(-6)}_{l'm'n'} = \frac{1}{2}\left[(\mu_l^2+\mu_m^2+\mu_n^2)^2 - 2\sum_{x=l,m,n}\mu_x^4\right]a_{lmn} \tag{4.146}$$

where the generalized coupling x is defined in Eq. 4.66. This equation still does not allow us to evaluate the LHS of Eq. (4.139) because it was derived only using the B-to-A relations and, thus, if applied to the LHS will merely reproduce the RHS of Eq. (4.139). We must find another route. Ideally, we will find a way of using completeness to perform the sum over the index j on the LHS of Eq. (4.139), which cannot be accomplished so long as all wavefunctions are differentiated as in $x^{(-6)}_{l'm'n'}$. Therefore, we can continue making progress by using integration by parts to remove the derivative from ψ_l in the integral $x^{(-6)}_{l'm'n'}$:

$$x^{(-6)}_{l'm'n'} = \frac{1}{\pi}\int d\varphi\ \varepsilon^{+2}(\partial_\varphi|\varphi|)(\partial_\varphi\psi_l)(\mathcal{D}\psi_m)(\mathcal{D}\psi_n) \tag{4.147}$$

$$= -\frac{1}{\pi}\int d\varphi\ \psi_l\partial_\varphi\left[\varepsilon^{+2}(\partial_\varphi|\varphi|)(\mathcal{D}\psi_m)(\mathcal{D}\psi_n)\right] \tag{4.148}$$

The distribution of ∂_φ on the quantity in square brackets will yield, among other terms,

$$\frac{1}{\pi}\int d\varphi\ \varepsilon^{+2}(\partial_\varphi^2|\varphi|)\psi_l(\mathcal{D}\psi_m)(\mathcal{D}\psi_n) \tag{4.149}$$

which vanishes because $(\partial_\varphi^2|\varphi|) = 2(\delta_0 - \delta_{\pi r_c})$ and $(\partial_\varphi\psi_n) = 0$ at the branes. Keeping in this mind, the remaining terms are

$$x^{(-6)}_{l'm'n'} = (-2kr_c)x^{(-6)}_{lm'n'} + \mu_m^2 x^{(-4)}_{lmn'} + \mu_n^2 x^{(-4)}_{lm'n} \tag{4.150}$$

such that

$$\mu_l^2 b_{lm'n'} = -12(kr_c)^2 x^{(-6)}_{lm'n'} + 6(kr_c)\left[\mu_m^2 x^{(-4)}_{lmn'} + \mu_n^2 x^{(-4)}_{lm'n}\right] - \mu_m^2 b_{l'mn'} - \mu_n^2 b_{l'm'n} \quad (4.151)$$

All terms on the RHS of this equation either lack derivatives on ψ_l or have fewer than four powers of μ_l after applying the B-to-A formulas and hence will be able to be handled via existing sum rules. Therefore, we proceed:

$$\sum_j \mu_j^2 b_{k'l'j} b_{jm'n'} = \sum_j b_{k'l'j}\Bigg[-12(kr_c)^2 x^{(-6)}_{jm'n'} + 6(kr_c)\left[\mu_m^2 x^{(-4)}_{jmn'} + \mu_n^2 x^{(-4)}_{jm'n}\right] - \mu_m^2 b_{j'mn'} - \mu_n^2 b_{j'm'n}\Bigg] \quad (4.152)$$

$$= -12(kr_c)^2 d_{klmn} + 6(kr_c)\left[\mu_m^2 x^{(-6)}_{k'l'mn'} + \mu_n^2 x^{(-6)}_{k'l'm'n}\right] - \mu_m^2 \sum_j b_{k'l'j} b_{j'mn'} - \mu_n^2 \sum_j b_{k'l'j} b_{j'm'n} \quad (4.153)$$

where

$$d_{klmn} \equiv x^{(-8)}_{k'l'm'n'} = \frac{1}{\pi}\int d\varphi\ (\mathcal{D}\psi_k)(\mathcal{D}\psi_l)(\mathcal{D}\psi_m)(\mathcal{D}\psi_n)\varepsilon^{+8} \quad (4.154)$$

Next, we must determine how to rewrite d_{klmn}, $x^{(-6)}_{k'l'm'n}$, and $x^{(-6)}_{k'l'mn'}$ in terms of a_{klmn} and c_{klmn}. The necessary equation for the latter two quantities can be derived by absorbing a factor of μ^2 into the quartic B-type coupling:

$$\mu_n^2 b_{k'l'mn} = \frac{1}{\pi}\int d\varphi\ \varepsilon^{-2}(\partial_\varphi\psi_k)(\partial_\varphi\psi_l)\psi_m \left[\mu_n^2 \varepsilon^{-2}\psi_n\right] \quad (4.155)$$

$$= \frac{1}{\pi}\int d\varphi\ \varepsilon^{-4}(\partial_\varphi\psi_n)\,\partial_\varphi\left[\varepsilon^{-2}(\partial_\varphi\psi_k)(\partial_\varphi\psi_l)\psi_m\right] \quad (4.156)$$

$$= \frac{1}{\pi}\int d\varphi\ \varepsilon^{-4}(\partial_\varphi\psi_n)\,\partial_\varphi\left[\varepsilon^{+6}(\mathcal{D}\psi_k)(\mathcal{D}\psi_l)\psi_m\right] \quad (4.157)$$

$$= \frac{1}{\pi}\int d\varphi\ \varepsilon^{+2}(\partial_\varphi\psi_n)\Big[+6(kr_c)(\partial_\varphi|\varphi|)(\mathcal{D}\psi_k)(\mathcal{D}\psi_l)\psi_m - \mu_k^2\varepsilon^{-2}\psi_k(\mathcal{D}\psi_l)\psi_m - \mu_l^2\varepsilon^{-2}(\mathcal{D}\psi_k)\psi_l\psi_m + \varepsilon^{+4}(\mathcal{D}\psi_k)(\mathcal{D}\psi_l)(\mathcal{D}\psi_m)\Big] \quad (4.158)$$

$$= \frac{6kr_c}{\pi}\int d\varphi\ (\partial_\varphi|\varphi|)\varepsilon^{+6}(\mathcal{D}\psi_k)(\mathcal{D}\psi_l)\psi_m(\mathcal{D}\psi_n) - \mu_k^2 b_{kl'mn'} - \mu_l^2 b_{k'lmn'} + \frac{1}{\pi}\int d\varphi\ \varepsilon^{+10}(\mathcal{D}\psi_k)(\mathcal{D}\psi_l)(\mathcal{D}\psi_m)(\mathcal{D}\psi_n) \quad (4.159)$$

which implies

$$\mu_k^2 b_{kl'mn'} + \mu_l^2 b_{k'lmn'} + \mu_n^2 b_{k'l'mn} = 6(kr_c)x^{(-6)}_{k'l'mn'} + c_{klmn} \quad (4.160)$$

Note the special role of the label m on both sides of this equation. The B-to-A formulas then yield, after some relabeling (i.e. $m \leftrightarrow n$),

$$6(kr_c)x^{(-6)}_{k'l'm'n} = \frac{1}{6}\left[-2(\mu_k^2+\mu_l^2+\mu_m^2)^2 + \sum_{x=k,l,m} \mu_x^2(3\mu_x^2+\mu_n^2)\right] a_{klmn} + c_{klmn} \quad (4.161)$$

which expresses $x^{(-6)}_{k'l'm'n}$ in terms of a_{klmn} and c_{klmn} as desired. Now to do the same for d_{klmn}. Consider absorbing μ_n^2 into $x^{(-6)}_{k'l'm'n}$:

$$\mu_n^2 x^{(-6)}_{k'l'm'n} = \frac{1}{\pi}\int d\varphi\ (\partial_\varphi|\varphi|)\varepsilon^{+8}(\mathcal{D}\psi_k)(\mathcal{D}\psi_l)(\mathcal{D}\psi_m)\left[\mu_n^2\varepsilon^{-2}\psi_n\right] \quad (4.162)$$

$$= \frac{1}{\pi}\int d\varphi\ (\mathcal{D}\psi_n)\partial_\varphi\left[(\partial_\varphi|\varphi|)\varepsilon^{+8}(\mathcal{D}\psi_k)(\mathcal{D}\psi_l)(\mathcal{D}\psi_m)\right] \quad (4.163)$$

Because $(\partial_\varphi^2|\varphi|) = 2(\delta_0 - \delta_{\pi r_c})$ and $(\partial_\varphi\psi_n) = 0$ at the branes,

$$\frac{1}{\pi}\int d\varphi\ (\partial_\varphi^2|\varphi|)\varepsilon^{+8}(\mathcal{D}\psi_k)(\mathcal{D}\psi_l)(\mathcal{D}\psi_m)(\mathcal{D}\psi_n) = 0 \quad (4.164)$$

such that Eq. (4.163) implies, after multiplying both sides by kr_c,

$$(kr_c)\left[\mu_k^2 x^{(-6)}_{kl'm'n'} + \mu_l^2 x^{(-6)}_{k'lm'n'} + \mu_m^2 x^{(-6)}_{k'l'mn'} + \mu_n^2 x^{(-6)}_{k'l'm'n}\right] = 8(kr_c)^2 d_{klmn} \quad (4.165)$$

We multiply by kr_c to enable the use of Eq. (4.161) on every term of the LHS, with which we obtain

$$(kr_c)^2 d_{klmn} = -\frac{1}{432}\left[(\vec{\mu}^{\,2})^3 - \sum_{x=k,l,m,n}\left(\mu_x^6 + 24\frac{\mu_k^2\mu_l^2\mu_m^2\mu_n^2}{\mu_x^2}\right)\right]a_{klmn} - \frac{1}{48}\vec{\mu}^{\,2}c_{klmn} \quad (4.166)$$

which expresses d_{klmn} in terms of a_{klmn} and c_{klmn} as desired. Now every term on the RHS of Eq. (4.153) can be expressed in terms of c_{klmn} or a_{klmn}.

Despite the symmetry of the LHS of Eq. (4.153), the expression that results from this process is not symmetric under $(k,l) \leftrightarrow (m,n)$: we can get a different expression by instead absorbing μ_j^2 into $b_{k'l'j}$. Because the end product of these different procedures must be equivalent, their difference must vanish. This yields a means of writing c_{klmn} entirely in terms of a_{klmn} whenever $\mu_k^2+\mu_l^2-\mu_m^2-\mu_n^2$ is nonzero:

$$(\mu_k^2+\mu_l^2-\mu_m^2-\mu_n^2)c_{klmn}$$
$$= \frac{2}{3}\left[\mu_m^2\mu_n^2\left(\mu_m^2+\mu_n^2-2(\mu_k^2+\mu_l^2)\right) - \mu_k^2\mu_l^2\left(\mu_k^2+\mu_l^2-2(\mu_m^2+\mu_n^2)\right)\right]a_{klmn} \quad (4.167)$$

In the elastic case, all external masses are equal ($\mu_k^2=\mu_l^2=\mu_m^2=\mu_n^2$) and both sides vanish, such that this equation provides no information on c_{klmn}. This does not affect the present derivation, for which we instead create a balanced version of the sum,

$$\sum_j \mu_j^2 b_{k'l'j} b_{jm'n'} = \frac{1}{2}\left[\sum_j \left(\mu_j^2 b_{k'l'j}\right) b_{jm'n'} + \sum_j b_{k'l'j}\left(\mu_j^2 b_{jm'n'}\right)\right]$$

$$= -12(kr_c)^2 d_{klmn} + 3(kr_c)\left[\mu_k^2 x^{(-6)}_{kl'm'n'} + \cdots + \mu_n^2 x^{(-6)}_{k'l'm'n}\right] \quad (4.168)$$

$$- \frac{1}{2}\left[\mu_k^2 \sum_j b_{kl'j'} b_{jm'n'} + \mu_l^2 \sum_j b_{k'lj'} b_{jm'n'}\right.$$

$$\left. + \mu_m^2 \sum_j b_{k'l'j} b_{j'mn'} + \mu_n^2 \sum_j b_{k'l'j} b_{j'm'n}\right] \quad (4.169)$$

Finally, we apply the B-to-A formulas and Eqs. (4.161) and (4.166) to Eq. (4.169), set the result equal to Eq. (4.139), and solve for the unknown sum to derive the last desired sum rule:

$$\sum_j \mu_j^6 a_{jkl} a_{jmn} = 5\vec{\mu}^{\,2} c_{klmn} - \frac{1}{9}\left[6(\mu_k^4 + \mu_l^4)(\mu_m^2 + \mu_n^2) + 6(\mu_k^2 + \mu_l^2)(\mu_m^4 + \mu_n^4)\right.$$

$$\left. - 4(\mu_k^2 + \mu_l^2)^3 - 4(\mu_m^2 + \mu_n^2)^3 + (\mu_k^6 + \mu_l^6 + \mu_m^6 + \mu_n^6)\right] a_{klmn} \quad (4.170)$$

In the next subsection, we summarize the principal results of this section.

4.4.5 Summary of Sum Rules (Inleastic)

All B-type couplings can be eliminated in favor of A-type couplings via B-to-A formulas

$$b_{l'm'n} = \frac{1}{2}\left[\mu_l^2 + \mu_m^2 - \mu_n^2\right] a_{lmn} \qquad b_{k'l'mn} = \frac{1}{6}\left[2\mu_k^2 + 2\mu_l^2 - \mu_m^2 - \mu_n^2\right] a_{klmn} \quad (4.171)$$

Applying the B-to-A formulas reduces the number of sums relevant to the cancellations we examine in the next chapter. These sums are

$$\sum_j a_{jkl} a_{jmn} = a_{klmn} \quad (4.172)$$

$$\sum_j \mu_j^2 a_{jkl} a_{jmn} = \frac{1}{3}\vec{\mu}^{\,2} a_{klmn} \quad (4.173)$$

$$\sum_j \mu_j^4 a_{jkl} a_{jmn} = 4c_{klmn} + \left[\frac{1}{3}(\vec{\mu}^{\,2})^2 - (\mu_k^2 + \mu_l^2)(\mu_m^2 + \mu_n^2)\right] a_{klmn} \quad (4.174)$$

$$\sum_j \mu_j^6 a_{jkl} a_{jmn} = 5\vec{\mu}^{\,2} c_{klmn} - \frac{1}{9}\left[6(\mu_k^4 + \mu_l^4)(\mu_m^2 + \mu_n^2) + 6(\mu_k^2 + \mu_l^2)(\mu_m^4 + \mu_n^4)\right.$$

$$\left. - 4(\mu_k^2 + \mu_l^2)^3 - 4(\mu_m^2 + \mu_n^2)^3 + (\mu_k^6 + \mu_l^6 + \mu_m^6 + \mu_n^6)\right] a_{klmn} \quad (4.175)$$

where $\vec{\mu}^{\,2} \equiv \mu_k^2 + \mu_l^2 + \mu_m^2 + \mu_n^2$, and

$$c_{klmn} \equiv \frac{1}{\pi} \int d\varphi \; \varepsilon^{-6} (\partial_\varphi \psi_k)(\partial_\varphi \psi_l)(\partial_\varphi \psi_m)(\partial_\varphi \psi_n) \tag{4.176}$$

The last two sum rules can be combined as to cancel all factors of c_{klmn}, and thereby yield

$$\sum_j \left[\mu_j^2 - \frac{5}{4}\vec{\mu}^{\,2}\right] \mu_j^2 a_{jkl} a_{jmn} = -\frac{1}{36}\Big[24(\mu_k^4 + \mu_l^4)(\mu_m^2 + \mu_n^2) + 24(\mu_k^2 + \mu_l^2)(\mu_m^4 + \mu_n^4) \\ - (\mu_k^2 + \mu_l^2)^3 - (\mu_m^2 + \mu_n^2)^3 + 4(\mu_k^6 + \mu_l^6 + \mu_m^6 + \mu_n^6)\Big] a_{klmn} \tag{4.177}$$

These equations extend and generalize the sum rules derived in [18].

4.4.6 Summary of Sum Rules (Elastic)

We are particularly interested in the elastic massive KK mode scattering process, wherein $k = l = m = n \neq 0$ and relations of the the previous subsection simplify. The relevant B-to-A formulas are

$$b_{n'n'j} = \frac{1}{2}\left[\mu_n^2 - \mu_j^2\right] a_{nnj} \qquad b_{j'n'n} = \frac{1}{2}\mu_j^2 a_{nnj} \qquad b_{n'n'nn} = \frac{1}{3}\mu_n^2 a_{nnnn} \tag{4.178}$$

whereas the sum rules become

$$\sum_j a_{jnn}^2 = a_{nnnn} \tag{4.179}$$

$$\sum_j \mu_j^2 a_{jnn}^2 = \frac{4}{3}\mu_m^2 a_{nnnn} \tag{4.180}$$

$$\sum_j \mu_j^4 a_{jnn}^2 = 4c_{nnnn} + \frac{4}{3}\mu_n^4 a_{nnnn} \tag{4.181}$$

$$\sum_j \mu_j^6 a_{jnn}^2 = 20c_{nnnn} + \frac{4}{3}\mu_n^6 a_{nnnn} \tag{4.182}$$

with the last two expressions combining to yield

$$\sum_j \left[\mu_j^2 - \frac{5}{4}\vec{\mu}^{\,2}\right] \mu_j^2 a_{jkl} a_{jmn} = -\frac{16}{3}\mu_n^6 a_{nnnn} \tag{4.183}$$

We now have all the elements necessary to begin calculating and analyzing amplitudes, which is the focus on the next chapter.

Chapter 5

Massive Spin-2 KK Mode Scattering in the RS1 Model

5.1 Chapter Summary

We will now apply the original material from chapters 3 and 4 to achieve the main theoretical results of this dissertation. In the last chapter, we used weak field expansion (WFE) and Kaluza-Klein (KK) decomposition to rewrite the 5D fields of the 5D RS1 model in terms of the following 4D field content: a massless spin-2 graviton $\hat{h}^{(0)}_{\mu\nu}$, a tower of massive spin-2 states $\hat{h}^{(n)}_{\mu\nu}$ with KK numbers $n \in \{1, 2, \dots\}$, and a massless spin-0 radion $\hat{r}^{(0)}$. We also derived the interactions between these 4D states by integrating the 5D WFE RS1 Lagrangian (which we derived in Chapter 3 and summarized in Eqs. (3.164)-(3.186)) over the extra dimension, thereby obtaining the 4D effective RS1 Lagrangian $\mathcal{L}^{(\text{eff})}_{4\text{D}}$ up to quartic order in the fields. The 5D and 4D effective theories were found to be related via the 5D-to-4D formula, Eq. (4.65):

$$\mathcal{L}^{(\text{RS,eff})}_{h^H r^R} = \left[\frac{\kappa}{\sqrt{\pi r_c}}\right]^{(H+R-2)} \sum_{\vec{n}=\vec{0}}^{+\infty} \left\{ a_{(R|\vec{n})} \cdot \mathcal{X}_{(\vec{n})}\Big[\,\overline{\mathcal{L}}_{A:h^H r^R}\Big] + b_{(R|\vec{n})} \cdot \mathcal{X}_{(\vec{n})}\Big[\,\overline{\mathcal{L}}_{B:h^H r^R}\Big] \right\} . \tag{5.1}$$

where $a_{(R|\vec{n})}$ and $b_{(R|\vec{n})}$ are integrals of products of KK wavefunctions which depend on the number of radions R and the KK numbers $\vec{n} = (n_1, \cdots, n_H)$ of the H spin-2 modes in a given term. Specifically, these integrals were defined in Eqs. (4.56) and (4.60) (and later generalized in Eq. (4.66)):

$$a_{(R|\vec{n})} \equiv a_{r\cdots r n_1 \cdots n_H} \equiv \frac{1}{\pi} e^{-R\pi k r_c} \int_{-\pi}^{+\pi} d\varphi \;\, \varepsilon^{2(R-1)} \psi_{n_1} \cdots \psi_{n_H} \left[\psi_0\right]^R \;, \tag{5.2}$$

$$\begin{aligned} b_{(R|\vec{n})} &\equiv b_{r\cdots r n_1' n_2' n_3 \cdots n_H} \;, \\ &\equiv \frac{1}{\pi} e^{-R\pi k r_c} \int_{-\pi}^{+\pi} d\varphi \;\, \varepsilon^{2(R-2)} (\partial_\varphi \psi_{n_1})(\partial_\varphi \psi_{n_2}) \psi_{n_3} \cdots \psi_{n_H} \left[\psi_0\right]^R \;, \end{aligned} \tag{5.3}$$

which define the A-type and B-type couplings respectively. Using the fact that the wavefunctions ψ_n satisfy a Sturm-Liouville problem, Eq. (4.36),

$$\partial_\varphi \left[\varepsilon^{-4} (\partial_\varphi \psi_n)\right] = -\mu_n^2 \varepsilon^{-2} \psi_n \tag{5.4}$$

with $(\partial_\varphi \psi_n) = 0$ at branes ($\varphi \in \{0, \pi\}$), various relations between the couplings and mass spectrum $\{\mu_n\} = \{m_n r_c\}$ were derived (Eqs. (4.171)-(4.183)). Chief among these were the ability to rewrite all B-type couplings in terms of A-type couplings, Eq. (4.171),

$$b_{l'm'n} = \frac{1}{2}\left[\mu_l^2 + \mu_m^2 - \mu_n^2\right] a_{lmn} \qquad b_{k'l'mn} = \frac{1}{6}\left[2\mu_k^2 + 2\mu_l^2 - \mu_m^2 - \mu_n^2\right] a_{klmn} \tag{5.5}$$

and certain elastic sum rules

$$\sum_j a_{jnn}^2 = a_{nnnn} \tag{5.6}$$

$$\sum_j \mu_j^2 a_{jnn}^2 = \frac{4}{3}\mu_m^2 a_{nnnn} \tag{5.7}$$

$$\sum_j \mu_j^4 a_{jnn}^2 = 4c_{nnnn} + \frac{4}{3}\mu_n^4 a_{nnnn} \tag{5.8}$$

$$\sum_j \mu_j^6 a_{jnn}^2 = 20c_{nnnn} + \frac{4}{3}\mu_n^6 a_{nnnn} \tag{5.9}$$

where $c_{klmn} \equiv \frac{1}{\pi}\int d\varphi\ \varepsilon^{-6}(\partial_\varphi\psi_k)(\partial_\varphi\psi_l)(\partial_\varphi\psi_m)(\partial_\varphi\psi_n)$, with the last two expressions combining to yield

$$\sum_j \left[\mu_j^2 - \frac{5}{4}\vec{\mu}^{\,2}\right]\mu_j^2 a_{jkl} a_{jmn} = -\frac{16}{3}\mu_n^6 a_{nnnn} \tag{5.10}$$

This chapter uses all of these results to calculate and then analyze matrix elements.

Recall our analogy between the Standard Model and the RS1 model from Chapter 1, which we originally laid out in Table 1.5 and have repeated in Table 5.1 for convenience. In this chapter, we finally confirm several elements of this table and draw the major conclusions of this dissertation:[1]

- Scattering of massive spin-2 KK modes in the RS1 model has a matrix element that grows like $\mathcal{O}(s)$ at large energies, regardless of helicity combination. Thus, scattering of the massive spin-2 KK modes behaves just like the scattering of 4D gravitons at high energies.

- Truncating the tower of massive spin-2 states (i.e. ignoring KK modes with KK numbers greater than some value N) generates a matrix element that grows like $\mathcal{O}(s^5)$, which replicates the bad high-energy behavior of, for example, massive spin-2 scattering in Fierz-Pauli gravity.

[1] These conclusions have been published across several papers: the high-energy scaling behaviors of the helicity-zero spin-2 KK mode scattering matrix element and each of its channels were published in [16]; the four sum rules which make those scaling behaviors possible for the elastic process were published in [17], which also included proofs for two of the sum rules; and all of these results were then elaborated on and generalized in [18]. This most recent paper also provides explicit versions of the 5D WFE and 4D effective RS1 Lagrangians (which we recounted and updated in Chapters 3 and 4), proves another sum rule (which we generalized in Chapter 4), analyzes how truncation of the KK tower affects the total matrix element, and calculates the strong-coupling scale from the 4D effective theory.

- Eliminating the radion from the matrix element calculation causes the matrix element to grow like $\mathcal{O}(s^3)$, which still reflects the explicit breaking of the underlying symmetry group but is more mild energy growth than the growth we attained by eliminating massive KK modes.

The rest of the chapter proceeds as follows:

- Section 5.2 establishes the definitions and conventions necessary to calculate the tree-level 2-to-2 scattering matrix element for massive spin-2 KK modes in the center-of-momentum frame. The section ends with some considerations regarding numerical analysis of the RS1 model.

- Section 5.3 considers the scattering of helicity-zero massive spin-2 states in the 5D orbifolded torus (5DOT) model, the limit of the RS1 model in which kr_c vanishes. The 5DOT model exhibits discrete KK momentum conservation: this allows all coupling integrals to be evaluated analytically and ensures only a finite number of diagrams contribute to the matrix element. The helicity-zero matrix element is found to grow like $\mathcal{O}(s)$ for any combination of external KK numbers that conserves discrete KK momentum (and vanishes otherwise). The helicity-zero process $(1,4) \to (2,3)$ lacks any massless intermediate states because of KK momentum conservation and thus the partial wave amplitudes of its matrix element can be calculated without running into massless poles. We calculate its leading partial wave amplitude a^0 and find via the partial wave amplitude constraints that the 5DOT strong-coupling scale is $\Lambda^{(\text{5DOT})}_{\text{strong}} = \sqrt{4\pi}\, M_{\text{Pl}}$.

- Section 5.4 considers the elastic scattering of massive spin-2 states in the RS1 model in which all external KK modes have equal KK number n, beginning with helicity-zero elastic scattering. The $\mathcal{O}(s^\sigma)$ contributions to the helicity-zero matrix element are demonstrated to cancel via certain sum rules for $\sigma = 5$, 4, 3, and finally 2. Of the sum rules obtained, only one linear combination was not proved in the previous chapter: this combination involves the radion coupling, and its validity is instead demonstrated numerically. An analytic expression for the residual $\mathcal{O}(s)$ amplitude is provided. Lastly, it is noted that the aforementioned helicity-zero sum rules are sufficient to ensure all elastic massive spin-2 KK mode scattering matrix elements grow at most like $\mathcal{O}(s)$, regardless of helicity combination.

- Section 5.5 is devoted to several numerical investigations. Subsection 5.5.1 demonstrates cancellations down to $\mathcal{O}(s)$ in helicity-zero inelastic scattering matrix elements. Subsection 5.5.1 investigates how truncation affects the accuracy of the matrix element and its leading $\mathcal{O}(s^\sigma)$ contributions ($\sigma \in \{1,2,3,4,5\}$) relative to the full matrix element without truncation. Subsection 5.5.3 calculates the RS1 strong-coupling scale $\Lambda^{(\text{RS1})}_{\text{strong}}$ using the 4D effective RS1 theory. Massless poles in RS1 matrix elements are avoided by comparing the leading $\mathcal{O}(s)$ matrix element growth in the RS1 model to the exactly calculable equivalent in the 5DOT model. This yields $\Lambda^{(\text{RS1})}_{\text{strong}} \sim \Lambda_\pi$ as expected based on the 5D RS1 theory.

This completes the major results this dissertation intended to present. The next chapter provides a brief conclusion that summarizes our original results as well as directions for future work.

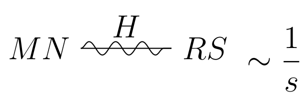
5.2 Motivation and Definitions

5.2.1 Restating the Problem

From the perspective of the 5D Lagrangian, the only excitation in the RS1 model is a massless 5D graviton H, which (when using the appropriate five-dimensional generalization of the helicity operator) has five helicity eigenstates. Because each term of $\mathcal{L}_{5\text{D}}$ contains two derivatives, each interaction vertex contains at most two powers of 4D momentum per term. Consequently, the cubic and quartic couplings grow like $\mathcal{O}(s)$ at high energies

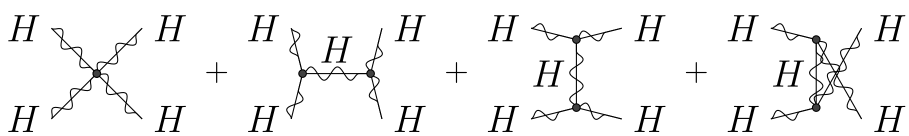
(5.11)

whereas the propagator falls like $\mathcal{O}(s^{-1})$



(5.12)

and the external polarizations are independent of s. The total tree-level matrix element for 2-to-2 scattering of 5D gravitons is the sum of four diagrams:



(5.13)

By combining the existing scaling arguments for each piece of each diagram, we find the overall matrix element must grow at most like $\mathcal{O}(s)$. We can arrive at this same conclusion by considering each graviton at energies so large that it can be localized with a width significantly less than the compactification radius r_c and inverse warping parameter $1/k$. At these energies, the only dimensionful parameter remaining is the coupling strength $\kappa_{5\text{D}}$. Therefore, because a 5D 2-to-2 scattering matrix element has units of inverse-energy and $[\kappa_{5\text{D}}] = [\text{Energy}]^{-3/2}$, the 5D matrix element must scale at high energies like

$$\mathcal{M}_{HH\to HH} \sim \kappa_{5\text{D}}^2 s \tag{5.14}$$

up to dimensionless multiplicative constants. This scaling provides a strict constraint on the high-energy behavior of the 4D matrix elements, which we now consider.

Consider the same argument from the 4D perspective. Instead of perturbing the metric G to yield a 5D graviton field $H_{MN}(x,y)$, it is perturbed by 5D fields $h_{\mu\nu}(x,y)$ and $r(x)$ which transform covariantly under the 4D Lorentz group. As detailed in Section 4.3, $h_{\mu\nu}$

	Standard Model	**Randall-Sundrum 1**
The fundamental symmetry group...	$\mathbf{SU(2)_W \times U(1)_Y}$	5D diffeomorphisms
... w/ unitarity-violation scale...	N/A	$\Lambda_\pi = M_{\rm Pl}\, e^{-krc\pi}$
... and gauged by the...	electroweak bosons	5D RS1 graviton
... is spontaneously broken by...	the Higgs vev	background geometry
... to a new symmetry group...	$\mathbf{U(1)_Q}$	4D diffeomorphisms*
... gauged by the...	photon, γ	4D graviton, $h^{(0)}$
... resulting in a spin-0 state...	Higgs boson, H	radion, $r^{(0)}$
... as well as massive states built from fund. gauge bosons...	W-bosons, $W^\pm$ and Z-boson, Z	spin-2 KK modes, $h^{(n)}$ for $n \in \{1, 2, \dots\}$
The 2-to-2 gauge boson process...	$\gamma\gamma \to \gamma\gamma$	$h^{(0)}h^{(0)} \to h^{(0)}h^{(0)}$
... has $\mathcal{M}$ w/ high-energy growth $\sim$	$\mathcal{O}(s^0)$	$\mathcal{O}(s)$
... or, if naively given mass, ...	$\mathcal{O}(s^2)$	$\mathcal{O}(s^5)$
... yet 2-to-2 massive state process where mass arises via sym. break...	$W^+W^- \to W^+W^-$	$h^{(n_1)}h^{(n_2)} \to h^{(n_3)}h^{(n_4)}$
... has $\mathcal{M}$ w/ high-energy growth $\sim$	$\mathcal{O}(s^0)$	$\mathbf{O(s)}$
Breaking the fund. symmetry by...	elim. Z	KK tower truncation
... makes massive states scatter like naively-massive gauge bosons, $\mathcal{M} \sim$	$\mathcal{O}(s^2)$	$\mathbf{O(s^5)}$
Breaking the fund. symmetry by...	elim. the Higgs	elim. the radion
... makes massive states scatter $\sim$	$\mathcal{O}(s)$	$\mathbf{O(s^3)}$

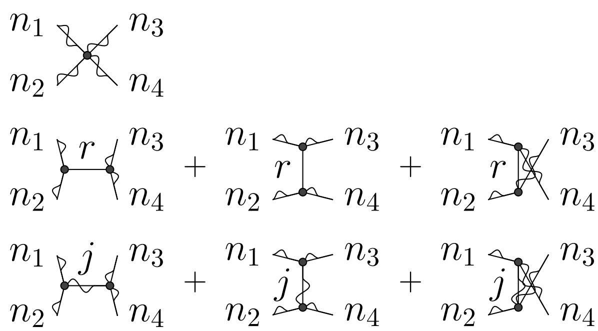
Table 5.1: The Standard Model (SM) and the Randall-Sundrum 1 (RS1) model share a chain of conceptual similarities with respect to the scattering of particles made massive by spontaneous symmetry breaking. The Mandelstam variable $s \equiv E^2$, where E is the incoming center-of-momentum energy. Original results presented in this dissertation are indicated in bold. (* - Technically, the new symmetry group is the Cartan subgroup of the 5D diffeomorphisms that contains the 4D diffeomorphisms.)

embeds a Kaluza-Klein (KK) tower of 4D spin-2 fields $h^{(n)}_{\mu\nu}(x)$, where $n = 0$ corresponds to the massless 4D graviton, and $r(x)$ embeds a massless 4D spin-0 state $r^{(0)}(x)$ called the radion. This dissertation focuses on tree-level 2-to-2 scattering of massive KK modes with (nonzero) KK indices n_1, n_2, n_3, and n_4. The matrix element $\mathcal{M}_{n_1 n_2 \to n_3 n_4}$ for this process is calculated from infinitely-many diagrams, which we categorize into subsets for ease of writing and discussion. All together, for any combination of external helicities,

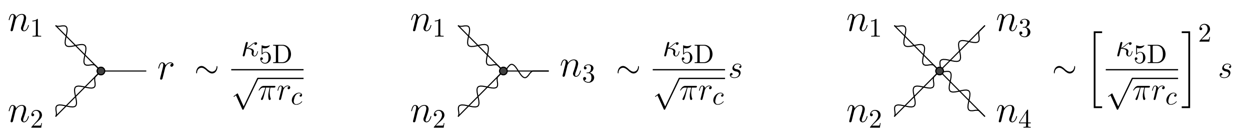
$$\mathcal{M}_{n_1 n_2 \to n_3 n_4} \equiv \mathcal{M}_{\rm c} + \mathcal{M}_{\rm r} + \sum_{j=0}^{+\infty} \mathcal{M}_j \ , \tag{5.15}$$

within which

$\mathcal{M}_{\rm c} \equiv$, $\mathcal{M}_{\rm r} \equiv$, $\mathcal{M}_j \equiv$ (5.16)

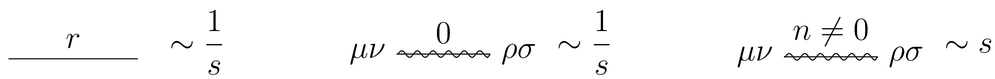


where subscript "c" denotes the contact diagram, "r" denotes the sum of diagrams mediated by the radion $\hat{r}^{(0)}$, and "j" denotes sum of diagrams mediated by the jth spin-2 KK mode $\hat{h}^{(j)}$. The relevant vertices scale like



(5.17)

where the hhr interaction does not grow in energy because the corresponding interaction Lagrangian (Eq. (4.78)) contains no 4D derivatives, and the relevant propagators scale like



according to Eqs. (2.344)-(2.353). The external massive spin-2 states can take on any one of five possible helicities $\lambda \in \{-2, -1, 0, 1, 2\}$, and are described by polarization tensors $\epsilon^{\mu\nu}_\lambda$ which have leading $\mathcal{O}(s^{2-|\lambda|})$ high-energy behavior (Eqs. (2.332) and (2.340)). In order to maximize energy growth, we focus on the helicity-zero process wherein $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. Under this assumption, if we combine these elements we find the diagrams seemingly scale like

$$\mathcal{M}_{j>0} \sim \mathcal{O}(s^7) \tag{5.18}$$

$$\mathcal{M}_0 \text{ and } \mathcal{M}_c \sim \mathcal{O}(s^5) \tag{5.19}$$

$$\mathcal{M}_r \sim \mathcal{O}(s^3) \tag{5.20}$$

such that naively we expect the matrix element $\mathcal{M}_{n_1 n_2 \to n_3 n_4}$ to grow like $\mathcal{O}(s^7)$ when all external massive spin-2 states have vanishing helicity. Explicit evaluation reveals that the scaling is slightly more mild in practice: per diagram,

$$\mathcal{M}_j \text{ and } \mathcal{M}_c \sim \mathcal{O}(s^5) \tag{5.21}$$

$$\mathcal{M}_r \sim \mathcal{O}(s^3) \tag{5.22}$$

where cancellations occur such that each diagram in $\mathcal{M}_{j>0}$ only grows like $\mathcal{O}(s^5)$. This suggests that $\mathcal{M}_{n_1 n_2 \to n_3 n_4}$ might grow as fast as $\mathcal{O}(s^5)$. However, such rapid energy growth would starkly contrast the high-energy growth of the 4D graviton, whose own 2-to-2 scattering matrix element only grows as fast as $\mathcal{O}(s)$. Inspired by the analogy with the Standard Model in Table 5.1, wherein the massive W-bosons scatter with the same high-energy behavior as photons due to the underlying electroweak symmetry $\mathbf{SU(2)_W} \times \mathbf{U(1)_Y}$, we expect that the matrix elements for scattering massive KK modes (which are generated by the 5D RS1 graviton just like the 4D graviton) should exhibit the same high-energy growth as graviton scattering, and indeed: this chapter demonstrates that cancellations occur between the diagrams in (5.15) which reduce the naive $\mathcal{O}(s^5)$ growth down to $\mathcal{O}(s)$ growth. These cancellations require precise relationships between the KK mode mass spectra and coupling integrals.

This chapter isolates those relationships and demonstrates they hold true in the 4D effective field theory. After this, the strong coupling scale Λ_π is calculated directly from the 4D effective theory, and the effects of KK mode truncation are investigated.

5.2.2 Definitions[2]

The preceding chapters detailed how to determine the vertices relevant to tree-level 2-to-2 scattering of massive spin-2 helicity eigenstates in the center-of-momentum frame. This subsection recounts the other diagrammatic pieces which go into calculating the diagrams relevant to those matrix elements. For scattering of nonzero KK modes $(n_1, n_2) \to (n_3, n_4)$ with helicities $(\lambda_1, \lambda_2) \to (\lambda_3, \lambda_4)$, we choose coordinates such that the initial particle pair have 4-momenta satisfying

$$p_1^\mu = (E_1, +|\vec{p}_i|\hat{z}) \qquad p_1^2 = m_{n_1}^2 \tag{5.23}$$

$$p_2^\mu = (E_2, -|\vec{p}_i|\hat{z}) \qquad p_2^2 = m_{n_2}^2 \tag{5.24}$$

and the final particle pair have 4-momenta satisfying

$$p_3^\mu = (E_3, +\vec{p}_f) \qquad p_3^2 = m_{n_3}^2 \tag{5.25}$$

$$p_4^\mu = (E_4, -\vec{p}_f) \qquad p_4^2 = m_{n_4}^2 \tag{5.26}$$

where $\vec{p}_f \equiv |\vec{p}_f|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. That is, the initial pair approach along the z-axis and the final pair separate along the line described by the angles (θ, ϕ). The helicity-λ

[2] This subsection was originally published as Subsection IV.A of [18], up to minor changes in wording.

spin-2 polarization tensor $\epsilon^{\mu\nu}_{\lambda}(p)$ for a particle with 4-momentum p is defined according to

$$\epsilon^{\mu\nu}_{\pm 2} = \epsilon^{\mu}_{\pm 1}\epsilon^{\nu}_{\pm 1} \ , \tag{5.27}$$

$$\epsilon^{\mu\nu}_{\pm 1} = \frac{1}{\sqrt{2}}\left[\epsilon^{\mu}_{\pm 1}\epsilon^{\nu}_{0} + \epsilon^{\mu}_{0}\epsilon^{\nu}_{\pm 1}\right] \tag{5.28}$$

$$\epsilon^{\mu\nu}_{0} = \frac{1}{\sqrt{6}}\left[\epsilon^{\mu}_{+1}\epsilon^{\nu}_{-1} + \epsilon^{\mu}_{-1}\epsilon^{\nu}_{+1} + 2\epsilon^{\mu}_{0}\epsilon^{\nu}_{0}\right] \ , \tag{5.29}$$

where ϵ^{μ}_{s} are the (particle-direction dependent) spin-1 polarization vectors

$$\epsilon^{\mu}_{\pm 1} = \pm\frac{e^{\pm i\phi}}{\sqrt{2}}\left(0, -c_\theta c_\phi \pm i s_\phi, -c_\theta s_\phi \mp i c_\phi, s_\theta\right) \ , \tag{5.30}$$

$$\epsilon^{\mu}_{0} = \frac{E}{m}\left(\sqrt{1-\frac{m^2}{E^2}}, \hat{p}\right) \ , \tag{5.31}$$

$(c_x, s_x) \equiv (\cos x, \sin x)$, and $\hat{p}$ is a unit vector in the direction of the momentum [31]. We use the Jacob-Wick second particle convention, which adds a phase $(-1)^{\lambda}$ to $\epsilon^{\mu\nu}_{\lambda}$ when the polarization tensor describes $h^{(n_2)}$ or $h^{(n_4)}$ [24]. Due to rotational invariance, we may set the azimuthal angle ϕ to 0 without loss of generality. Meanwhile, the propagators for virtual spin-0 and spin-2 particles of mass M and 4-momentum P are, respectively,

$$\text{———} \quad = \frac{i}{P^2 - M^2} \tag{5.32}$$

$$\mu\nu \ \text{~~~} \ \rho\sigma \ = \frac{iB^{\mu\nu,\rho\sigma}}{P^2 - M^2} \tag{5.33}$$

where we use the spin-2 propagator convention [31]

$$B^{\mu\nu,\rho\sigma} \equiv \frac{1}{2}\left[\overline{B}^{\mu\rho}\overline{B}^{\nu\sigma} + \overline{B}^{\nu\rho}\overline{B}^{\mu\sigma} - \frac{1}{3}(2+\delta_{0,M})\overline{B}^{\mu\nu}\overline{B}^{\rho\sigma}\right]$$

$$\overline{B}^{\alpha\beta}\Big|_{M=0} = \eta^{\alpha\beta} \qquad \overline{B}^{\alpha\beta}\Big|_{M\neq 0} \equiv \eta^{\alpha\beta} - \frac{P^\alpha P^\beta}{M^2} \tag{5.34}$$

and $\eta^{\mu\nu} = \mathrm{Diag}(+1,-1,-1,-1)$ is the flat 4D metric. The massless spin-2 propagator is derived in the de Donder gauge by adding a gauge-fixing term

$$\mathcal{L}_{gf} = -(\partial^\mu \hat{h}^{(0)}_{\mu\nu} - \tfrac{1}{2}\partial_\nu [\![\hat{h}^{(0)}]\!])^2 \tag{5.35}$$

to the Lagrangian. The Mandelstam variable $s \equiv (p_1+p_2)^2 = (E_1+E_2)^2$ provides a convenient frame-invariant measure of collision energy. The minimum value of s that is kinematically allowed equals $s_{\min} \equiv \max[(m_{n_1}+m_{n_2})^2, (m_{n_3}+m_{n_4})^2]$. When dealing with explicit full matrix elements, we will replace $s \in [s_{\min}, +\infty)$ with the unitless $\mathfrak{s} \in [0, +\infty)$ which is defined according to $s \equiv s_{\min}(1+\mathfrak{s})$.

As discussed in Subsection 5.2.1, any tree-level massive spin-2 scattering amplitude can be written as

$$\mathcal{M}_{n_1 n_2 \to n_3 n_4} \equiv \mathcal{M}_{\mathrm{c}} + \mathcal{M}_{\mathrm{r}} + \sum_{j=0}^{+\infty} \mathcal{M}_j \ , \tag{5.36}$$

where $\mathcal{M}_{n_1 n_2 \to n_3 n_4}$ will be abbreviated to $\mathcal{M}$ when the process can be understood from context, and we separate the contributions arising from contact interactions, radion exchange, and a sum over the exchanged intermediate KK states j (and where "0" labels the massless graviton). In practice, this sum cannot be completed in entirety and must instead be truncated. Therefore, we also define the truncated matrix element

$$\mathcal{M}^{[N]} \equiv \mathcal{M}_{\rm c} + \mathcal{M}_{\rm r} + \sum_{j=0}^{N} \mathcal{M}_j \ , \tag{5.37}$$

which includes the contact diagram, the radion-mediated diagrams, and all KK mode-mediated diagrams with intermediate KK number less than or equal to N.

We are concerned with the high-energy behavior of these matrix elements, and will therefore examine the high-energy behavior of each of the contributions discussed. Because the polarization tensors $\epsilon^{\mu\nu}_{\pm 1}$ introduce odd powers of energy, $\sqrt{s}$ is a more appropriate expansion parameter for generic helicity combinations. Thus, we series expand the diagrams and total matrix element in $\sqrt{s}$ and label the coefficients like so:

$$\mathcal{M}(s,\theta) \equiv \sum_{\sigma \in \frac{1}{2}\mathbb{Z}} \overline{\mathcal{M}}^{(\sigma)}(\theta) \cdot s^{\sigma} \tag{5.38}$$

and define $\mathcal{M}^{(\sigma)} \equiv \overline{\mathcal{M}}^{(\sigma)} \cdot s^{\sigma}$. In what follows, we demonstrate that $\mathcal{M}^{(\sigma)}$ vanishes for $\sigma > 1$ regardless of helicity combination and we present the residual linear term in s for helicity-zero elastic scattering.

5.2.3 Comments on Numerical Evaluation

The previous chapter detailed how to manipulate integrals of products of wavefunctions from a purely analytic perspective, so let us take a moment to consider the numerical perspective. In those cases where it is desirable to numerically evaluate matrix elements, it can be difficult to achieve a desired numerical accuracy for a variety of reasons. For example, the determination of the massive spin-2 KK mode spectrum via

$$\begin{aligned} &\left[2J_2 + \frac{\mu_n \varepsilon}{kr_c}(\partial J_2)\right]\Bigg|_{\varphi=\pi} \left[2Y_2 + \frac{\mu_n \varepsilon}{kr_c}(\partial Y_2)\right]\Bigg|_{\varphi=0} \\ &\quad - \left[2Y_2 + \frac{\mu_n \varepsilon}{kr_c}(\partial Y_2)\right]\Bigg|_{\varphi=\pi} \left[2J_2 + \frac{\mu_n \varepsilon}{kr_c}(\partial J_2)\right]\Bigg|_{\varphi=0} = 0 \end{aligned} \tag{5.39}$$

(which is Eq. (4.34) when $\nu = 2$) amounts to solving for the roots of the RHS to some desired accuracy. However, the spacing of those roots can vary dramatically depending on the value of kr_c, which means (depending on your root-solving method) there is the possibility to inadvertently skip roots. To avoid this, we can use our exact knowledge of the eigenvalue spectrum when $kr_c = 0$ (considered in the next section) and when kr_c is large (Subsection 4.3.5) to reparameterize Eq. (5.39) in terms of a variable wherein the roots are more evenly spaced. For this purpose, we use

$$\mu_n \equiv \frac{c_n}{n}\left[(kr_c)\,x_n\,e^{-kr_c\pi} + n\,e^{-kr_c\pi}\right] \tag{5.40}$$

and solve for c_n. Having obtained a sufficiently-accurate eigenvalue spectrum, it is then useful to rewrite ψ_n into the form

$$\psi_n = \frac{\varepsilon^2}{N_n}\left[b_{n2}^{(\text{den})}\,J_2\left(\frac{\mu_n\varepsilon}{kr_c}\right) - b_{n2}^{(\text{num})}\,Y_2\left(\frac{\mu_n\varepsilon}{kr_c}\right)\right] \tag{5.41}$$

rather than Eq. (4.32), where $b_{n2}^{(\text{num})}$ and $b_{n2}^{(\text{den})}$ indicate the numerator and denominator of Eq. (4.33) respectively. (The value of N_n must change to accommodate this new form but is still determined by Eq. (4.26).) This new form helps avoid numerical instability during the occasions when $b_{n2}^{(\text{den})}$ is close to zero. Furthermore, it is worthwhile to directly utilize the analytic form of derivatives wherever possible. Specifically, this means using

$$\partial J_\nu \equiv \frac{1}{2}\left[J_{\nu-1} - J_{\nu+1}\right] \qquad \partial Y_\nu \equiv \frac{1}{2}\left[Y_{\nu-1} - Y_{\nu+1}\right] \tag{5.42}$$

and

$$(\partial_\varphi\psi_n) = \frac{\varepsilon^3}{N_n}\mu_n\left[b_{n2}^{(\text{den})}\,J_1\left(\frac{\mu_n\varepsilon}{kr_c}\right) - b_{n2}^{(\text{num})}\,Y_1\left(\frac{\mu_n\varepsilon}{kr_c}\right)\right](\partial_\varphi|\varphi|) \tag{5.43}$$

which uses the same N_n derived when normalizing ψ_n in Eq. (5.41). These changes all help in gaining as much numerical accuracy as possible before calculating coupling integrals. As detailed in Section 4.3, interaction vertices in the effective theory are proportional to integrals of products of wavefunctions and their derivatives. Each wavefunction ψ_n oscillates through zero n times over the (half) domain $\varphi \in [0, \pi]$ and is exponentially distorted towards $\varphi = \pm\pi$ by an amount determined by the specific value of kr_c selected. Consequently, interaction vertices involving even relatively modest mode numbers ($n \sim 10$) generate integrands that are highly oscillatory. Those dramatic oscillations in the integrand lead to cancellations between large positive and large negative values in the integral, which can eliminate many significant digits worth of numerical confidence. The number of significant digits are retained following these cancellations depends on just how accurately the different maxima and minima cancel one-another, which varies dramatically from integral to integral. In this sense, the integrals required for investigations of the 4D effective RS1 model are numerically unstable. This results in a time-consuming feedback loop: the numerical accuracy of the spectrum and wavefunctions must be increased until the coupling integrals are sufficiently accurate, which can not be known until those integrals are attempted. Furthermore, because we are interesting in demonstrating cancellations between diagrams in the matrix element, we are often evaluating perturbative expressions in an attempt to "measure zero": because higher-order terms in those expansions contribute less than lower-order terms, their effects are only evident if the lower-order terms are evaluated to sufficient accuracy, further increasing the need for highly-accurate results. We can only be confident we have calculated all elements of the calculation to sufficient accuracy once all evidence of numerical noise is absent from certain cross-checks (such as the sum rules analytically proved in Section 4.4). Unfortunately, there seems to be no means of avoiding this time-consuming complication.

5.3 Elastic Scattering in the 5D Orbifolded Torus Model[3]

In this section, we begin our analysis of the scattering amplitudes of the massive spin-2 KK modes. As described above, the full tree-level scattering amplitudes will require summing over the exchange of all intermediate states, and we will find that the cancellations needed to reduce the growth of RS1 scattering amplitudes from $\mathcal{O}(s^5)$ to $\mathcal{O}(s)$ will only completely occur once all states are included. In the present section, we analyze KK mode scattering in a limit that only has finitely many nonzero diagrams per matrix element: the 5D Orbifolded Torus model.

The 5D Orbifolded Torus (5DOT) model is obtained by taking the limit of the RS1 metric Eq. (3.115) as kr_c vanishes, while simultaneously maintaining a nonzero finite first mass m_1 (or, equivalently, a nonzero finite r_c). Consequently, the 5DOT metric lacks explicit dependence on y,

$$G^{(\text{5DOT})}_{MN} = \begin{pmatrix} e^{\frac{-\kappa\hat{r}}{\sqrt{6}}}(\eta_{\mu\nu} + \kappa\hat{h}_{\mu\nu}) & 0 \\ 0 & -\left(1 + \frac{\kappa\hat{r}}{\sqrt{6}}\right)^2 \end{pmatrix} , \tag{5.44}$$

and as $kr_c \to 0$ the massive wavefunctions go from exponentially-distorted Bessel functions to simple cosines:

$$\psi_n = \begin{cases} \psi_0 = \frac{1}{\sqrt{2}} \\ \psi_n = -\cos(n|\varphi|) \quad 0 < n \in \mathbb{Z} \end{cases} \tag{5.45}$$

with masses given by $\mu_n = m_n r_c = n$ and 5D gravitational coupling $\kappa = \sqrt{2\pi r_c}\kappa_{4\text{D}} = \sqrt{8\pi r_c}/M_{\text{Pl}}$. In the absence of warp factors, the radion now couples diagonally and spin-2 interactions display discrete KK momentum conservation. Explicitly, an H-point vertex $\hat{h}^{(n_1)}\cdots\hat{h}^{(n_H)}$ in the 4D effective 5DOT model has vanishing coupling if there exists no choice of $c_i \in \{-1,+1\}$ such that $c_1 n_1 + \cdots + c_H n_H = 0$. For example, the three-point couplings $a_{n_1 n_2 n_3}$ and $b_{n_1' n_2' n_3}$ are nonzero only when $n_1 = |n_2 \pm n_3|$. Therefore, unlike when kr_c is nonzero, the 5DOT matrix element $\mathcal{M}^{(\text{5DOT})}$ for a process $(n_1, n_2) \to (n_3, n_4)$ consists of only finitely many nonzero diagrams.

For $(n,n) \to (n,n)$, the 5DOT matrix element arises from four types of diagrams:

$$\mathcal{M}^{(\text{5DOT})}_{(n,n)\to(n,n)} = \mathcal{M}_{\text{c}} + \mathcal{M}_{\text{r}} + \mathcal{M}_0 + \mathcal{M}_{2n} \ . \tag{5.46}$$

Using Eq. (4.67) and the 5DOT wavefunctions, we find:

$$\begin{aligned} n^2 a_{nnnn} &= 3b_{n'n'nn} = \frac{3}{4}n^2 , \\ n^2 a_{nn0} &= b_{n'n'0} = b_{n'n'r} = \frac{1}{\sqrt{2}}n^2 , \\ n^2 a_{nn(2n)} &= -b_{n'n'(2n)} = \frac{1}{2}b_{(2n)'n'n} = -\frac{1}{2}n^2 , \end{aligned} \tag{5.47}$$

[3] The first paragraph of this section originates from Section IV of [18]. The rest of this section's content was original published as Subsection IV.B of [18] up to minor changes.

	s^5	s^4	s^3	s^2
$\frac{1}{\kappa^2}\mathcal{M}_{\rm c}$	$-\frac{r_c^7[7+c_{2\theta}]s_\theta^2}{3072n^8\pi}$	$\frac{r_c^5[63-196c_{2\theta}+5c_\theta]}{9216n^6\pi}$	$\frac{r_c^3[-185+692c_{2\theta}+5c_\theta]}{4608n^4\pi}$	$-\frac{r_c[5+47c_{2\theta}]}{72n^2\pi}$
$\frac{1}{\kappa^2}\mathcal{M}_{2n}$	$\frac{r_c^7[7+c_{2\theta}]s_\theta^2}{9216n^8\pi}$	$\frac{r_c^5[-13+c_{2\theta}]s_\theta^2}{1152n^6\pi}$	$\frac{r_c^3[97+3c_{2\theta}]s_\theta^2}{1152n^4\pi}$	$\frac{r_c[-179+116c_{2\theta}-c_\theta]}{1152n^2\pi}$
$\frac{1}{\kappa^2}\mathcal{M}_0$	$\frac{r_c^7[7+c_{2\theta}]s_\theta^2}{4608n^8\pi}$	$\frac{r_c^5[-9+140c_{2\theta}-3c_\theta]}{9216n^6\pi}$	$\frac{r_c^3[15-270c_{2\theta}-c_\theta]}{2304n^4\pi}$	$\frac{r_c[175+624c_{2\theta}+c_\theta]}{1152n^2\pi}$
$\frac{1}{\kappa^2}\mathcal{M}_{\rm r}$	0	0	$-\frac{r_c^3 s_\theta^2}{64n^4\pi}$	$\frac{r_c[7+c_{2\theta}]}{96n^2\pi}$
Sum	0	0	0	0

Table 5.2: Cancellations in the $(n,n) \to (n,n)$ 5DOT amplitude, where $(c_\theta, s_\theta) = (\cos\theta, \sin\theta)$. Originally published in [16]

where here again the subscript "0" refers to the massless 4D graviton. We focus first on the scattering of helicity-zero states, which have the most divergent high-energy behavior (we return to consider other helicity combinations in Sec. 5.4.6). Figure 5.2 lists $\mathcal{M}_{\rm c}^{(\sigma)}$, $\mathcal{M}_{\rm r}^{(\sigma)}$, $\mathcal{M}_0^{(\sigma)}$, and $\mathcal{M}_{2n}^{(\sigma)}$ for $\sigma \geq 1$, and demonstrates how cancellations occur among them such that $\overline{\mathcal{M}}^{(\sigma)} = 0$ for $\sigma > 1$. The leading contribution in incoming energy is

$$\overline{\mathcal{M}}^{(1)} = \frac{3\kappa^2}{256\pi r_c}\,[7+\cos(2\theta)]^2 \csc^2\theta\,. \tag{5.48}$$

We report here the results of the full calculation, including subleading terms.

The complete (tree-level) matrix element for the elastic helicity-zero 5DOT process equals

$$\mathcal{M}^{(5\text{DOT})} = \frac{\kappa^2 n^2\,[P_0 + P_2 c_{2\theta} + P_4 c_{4\theta} + P_6 c_{6\theta}]\csc^2\theta}{256\pi r_c^3 \mathfrak{s}(\mathfrak{s}+1)(\mathfrak{s}^2+8\mathfrak{s}+8-\mathfrak{s}^2 c_{2\theta})}\,, \tag{5.49}$$

where

$$P_0 = 510\,\mathfrak{s}^5 + 3962\,\mathfrak{s}^4 + 8256\,\mathfrak{s}^3 + 7344\,\mathfrak{s}^2 + 3216\,\mathfrak{s} + 704\,, \tag{5.50}$$

$$P_2 = -429\,\mathfrak{s}^5 + 393\,\mathfrak{s}^4 + 3936\,\mathfrak{s}^3 + 5584\,\mathfrak{s}^2 + 3272\,\mathfrak{s} + 768\,, \tag{5.51}$$

$$P_4 = -78\,\mathfrak{s}^5 - 234\,\mathfrak{s}^4 + 192\,\mathfrak{s}^3 + 1552\,\mathfrak{s}^2 + 1776\,\mathfrak{s} + 576\,, \tag{5.52}$$

$$P_6 = -3\,\mathfrak{s}^5 - 25\,\mathfrak{s}^4 - 96\,\mathfrak{s}^3 - 144\,\mathfrak{s}^2 - 72\,\mathfrak{s}\,, \tag{5.53}$$

and $\mathfrak{s}$ is defined such that $s \equiv s_{\rm min}(1+\mathfrak{s})$ where in this case $s_{\rm min} = 4m_n^2 = 4n^2/r_c^2$.

For a generic helicity-zero 5DOT process $(n_1, n_2) \to (n_3, n_4)$, the leading high-energy contribution to the matrix element equals

$$\overline{\mathcal{M}}^{(1)} = \frac{\kappa^2}{256\pi r_c} x_{n_1 n_2 n_3 n_4}\,[7+\cos(2\theta)]^2 \csc^2\theta\,, \tag{5.54}$$

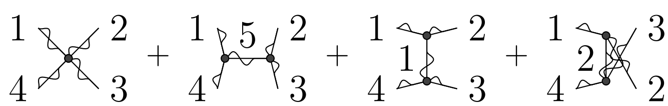
where x is fully symmetric in its indices, and satisfies

$$x_{aaaa} = 3, \quad x_{aabb} = 2, \quad \text{otherwise } x_{abcd} = 1 \ ,$$

when discrete KK momentum is conserved (and, of course, vanishes when the process does not conserve KK momentum).

The multiplicative $\csc^2\theta$ factor in Eq. (5.49) is indicative of t- and u-channel divergences from the exchange of the massless graviton and radion, which introduces divergences at $\theta = 0, \pi$. Such IR divergences prevent us from directly using a partial wave analysis to determine the strong coupling scale of this theory. In order to characterize the strong-coupling scale of this theory, we must instead investigate a nonelastic scattering channel for which KK momentum conservation implies that no massless states can contribute, $\mathcal{M}_0 = \mathcal{M}_r = 0$. (In this case, the $\csc^2\theta$ factor present in Eq. (5.54) is an artifact of the high-energy expansion and is absent from the full matrix element.)

Consider for example the helicity-zero 5DOT process $(1,4) \to (2,3)$. The total matrix element is computed from four diagrams



(5.55)

which together yield, after explicit computation,

$$\mathcal{M} = \frac{\kappa^2 \mathfrak{s}}{12800\pi r_c^3(\mathfrak{s}+1)^2 Q_+ Q_-} \sum_{i=0}^{4} Q_i c_{i\theta} \ , \tag{5.56}$$

where

$$Q_\pm = 25(\mathfrak{s}+1) \pm \left[3 + \sqrt{(25\,\mathfrak{s}+16)(25\,\mathfrak{s}+24)}\cos\theta\right] \ , \tag{5.57}$$

$$Q_0 = 15\left(2578125\,\mathfrak{s}^4 + 9437500\,\mathfrak{s}^3 + 12990000\,\mathfrak{s}^2 + 7971000\,\mathfrak{s} + 1840564\right) \ , \tag{5.58}$$

$$Q_1 = 72\sqrt{(25\,\mathfrak{s}+16)(25\,\mathfrak{s}+24)}(50\,\mathfrak{s}+43)(50\,\mathfrak{s}+47) \ , \tag{5.59}$$

$$Q_2 = 4\left(2734375\,\mathfrak{s}^4 + 11562500\,\mathfrak{s}^3 + 18047500\,\mathfrak{s}^2 + 12340500\,\mathfrak{s} + 3121692\right) \ , \tag{5.60}$$

$$Q_3 = 24\sqrt{(25\,\mathfrak{s}+16)(25\,\mathfrak{s}+24)}(50\,\mathfrak{s}+51)(50\mathfrak{s}+59) \ , \tag{5.61}$$

$$Q_4 = 390625\,\mathfrak{s}^4 + 2187500\,\mathfrak{s}^3 + 4360000\,\mathfrak{s}^2 + 3729000\,\mathfrak{s} + 1165956 \ , \tag{5.62}$$

and $s_{\min} = 25/r_c^2$. As expected, unlike the elastic 5DOT matrix element (5.49), the $(1,4) \to (2,3)$ 5DOT matrix element is finite at $\theta = 0, \pi$.

Given a 2-to-2 scattering process with helicities $(\lambda_1, \lambda_2) \to (\lambda_3, \lambda_4)$, the corresponding partial wave amplitudes a^J are defined as [24]

$$a^J = \frac{1}{32\pi^2} \int d\Omega \quad D^J_{\lambda_i \lambda_f}(\theta, \phi) \mathcal{M}(s, \theta, \phi) \ , \tag{5.63}$$

where $\lambda_i = \lambda_1 - \lambda_2$ and $\lambda_f = \lambda_3 - \lambda_4$, $d\Omega = d(\cos\theta)\, d\phi$, and the Wigner D functions $D^J_{\lambda_a\lambda_b}$ are normalized according to

$$\int d\Omega \quad D^J_{\lambda_a\lambda_b}(\theta,\phi)\cdot D^{J'*}_{\lambda'_a\lambda_b}(\theta,\phi) = \frac{4\pi}{2J+1}\cdot\delta_{JJ'}\cdot\delta_{\lambda_a\lambda'_a}\ . \tag{5.64}$$

Each partial wave amplitude is constrained by unitarity to satisfy

$$\sqrt{1-\frac{s_{\rm min}}{s}}\ \Re[a^J] \le \frac{1}{2}\ , \tag{5.65}$$

where $\Re[a^J]$ denotes the real part of a^J. The leading partial wave amplitude of the $(1,4)\to(2,3)$ helicity-zero 5DOT matrix element corresponds to $J=0$, and has leading term

$$a^0 \simeq \frac{s}{8\pi M^2_{\rm Pl}}\ \ln\left(\frac{s}{s_{\rm min}}\right)\ . \tag{5.66}$$

Hence, this matrix element violates unitarity when $\mathrm{Re}\, a^0 \simeq 1/2$, or equivalently when the value of $E \equiv \sqrt{s}$ is near or greater than $\Lambda^{(\rm 5DOT)}_{\rm strong} \equiv \sqrt{4\pi}M_{\rm Pl}$. Because $M_{\rm Pl}$ labels the reduced Planck mass, $\Lambda^{(\rm 5DOT)}_{\rm strong}$ is roughly the conventional Planck mass. We will use this inelastic calculation as a benchmark for estimating the strong-coupling scale associated with other processes.

We now consider the behavior of scattering amplitudes in the RS1 model.

5.4 Elastic Scattering in the Randall-Sundrum Model[4]

This section discusses the computation of the elastic scattering amplitudes of massive spin-2 KK modes in the RS1 model, for arbitrary values of the curvature of the internal space. For any nonzero curvature, every KK mode in the infinite tower contributes to each scattering process and the cancellation from $\mathcal{O}(s^5)$ to $\mathcal{O}(s)$ energy growth only occurs when all of these states are included. In the subsequent subsections, we apply the sum rules to determine the leading high-energy behavior of the amplitudes for two-body scattering of helicity-zero modes. Finally, Sec. 5.4.6 analyses the (milder) high-energy behavior of the scattering of nonzero-helicity modes of the massive spin-2 KK states.

5.4.1 Cancellations at $\mathcal{O}(s^5)$ in RS1

We will now go through the contributions to the elastic helicity-zero $(n,n)\to(n,n)$ scattering process in the RS1 model order by order in powers of s, and apply the sum rules derived in the previous chapter.

[4]The section description comes from Section V of [18]. The section content comprises Subsections V.B through V.G of [18] with some modification to update notation and utilize the new expressions of the sum rules from the previous chapter.

As described in Subsection 5.2.1, the contact diagram and spin-2-mediated diagrams individually diverge like $\mathcal{O}(s^5)$. After converting all $b_{\vec{n}}$ couplings into $a_{\vec{n}}$ couplings, their contributions to the elastic helicity-zero RS1 matrix element equal

$$\overline{\mathcal{M}}^{(5)}_c = -\frac{\kappa^2\, a_{nnnn}}{2304\,\pi r_c\, m_n^8}\,[7+\cos(2\theta)]\sin^2\theta\ , \tag{5.67}$$

$$\overline{\mathcal{M}}^{(5)}_j = \frac{\kappa^2\, a_{nnj}^2}{2304\,\pi r_c\, m_n^8}\,[7+\cos(2\theta)]\sin^2\theta\ , \tag{5.68}$$

such that they sum to

$$\overline{\mathcal{M}}^{(5)} = \frac{\kappa^2\,[7+\cos(2\theta)]\sin^2\theta}{2304\,\pi r_c\, m_n^8}\left\{\sum_{j=0}^{+\infty} a_{nnj}^2 - a_{nnnn}\right\}\ . \tag{5.69}$$

This vanishes via Eq. (4.179), which we will, henceforth, refer to as the $\mathcal{O}(s^5)$ sum rule.

5.4.2 Cancellations at $\mathcal{O}(s^4)$ in RS1

The $\mathcal{O}(s^4)$ contributions to the elastic helicity-zero RS1 matrix element equal

$$\overline{\mathcal{M}}^{(4)}_c = \frac{\kappa^2\, a_{nnnn}}{6912\,\pi r_c\, m_n^6}\,[63 - 196\cos(2\theta) + 5\cos(4\theta)]\ , \tag{5.70}$$

$$\overline{\mathcal{M}}^{(4)}_j = -\frac{\kappa^2\, a_{nnj}^2}{9216\,\pi r_c\, m_n^6}\left\{[7+\cos(2\theta)]^2\,\frac{m_j^2}{m_n^2} + 2\,[9 - 140\cos(2\theta) + 3\cos(4\theta)]\right\}. \tag{5.71}$$

Using the $\mathcal{O}(s^5)$ sum rule, $\overline{\mathcal{M}}^{(4)}$ equals

$$\overline{\mathcal{M}}^{(4)} = \frac{\kappa^2\,[7+\cos(2\theta)]^2}{9216\,\pi r_c\, m_n^6}\left\{\frac{4}{3}a_{nnnn} - \sum_j \frac{m_j^2}{m_n^2} a_{nnj}^2\right\}\ . \tag{5.72}$$

This vanishes via Eq. (4.180), which we shall refer to as the $\mathcal{O}(s^4)$ sum rule.

5.4.3 Cancellations at $\mathcal{O}(s^3)$ in RS1

Once the $\mathcal{O}(s^5)$ and $\mathcal{O}(s^4)$ contributions are cancelled, the radion-mediated diagrams, which diverge like $\mathcal{O}(s^3)$, become relevant to the leading behavior of the elastic helicity-zero RS1 matrix element. Furthermore, because of differences between the massless and massive spin-2 propagators, $\overline{\mathcal{M}}_0$ and $\overline{\mathcal{M}}_{j>0}$ differ from one another at this order (and lower). The full set of relevant contributions is therefore

$$\overline{\mathcal{M}}^{(3)}_{c} = \frac{\kappa^2\, a_{nnnn}}{3456\,\pi r_c\, m_n^4}\left[-185 + 692\cos(2\theta) + 5\cos(4\theta)\right]\ , \tag{5.73}$$

$$\overline{\mathcal{M}}^{(3)}_{r} = -\frac{\kappa^2}{32\,\pi r_c\, m_n^4}\left[\frac{b^2_{n'n'r}}{(m_n r_c)^4}\right]\sin^2\theta\ , \tag{5.74}$$

$$\overline{\mathcal{M}}^{(3)}_{0} = \frac{\kappa^2\, a^2_{nn0}}{1152\,\pi r_c\, m_n^4}\left[15 - 270\cos(2\theta) - \cos(4\theta)\right]\ , \tag{5.75}$$

$$\overline{\mathcal{M}}^{(3)}_{j>0} = \frac{\kappa^2\, a^2_{nnj}}{2304\,\pi r_c\, m_n^4}\Bigg\{5\left[1-\cos(2\theta)\right]\frac{m_j^4}{m_n^4} + \left[69 + 60\cos(2\theta) - \cos(4\theta)\right]\frac{m_j^2}{m_n^2} + 2\left[13 - 268\cos(2\theta) - \cos(4\theta)\right]\Bigg\}\ , \tag{5.76}$$

After applying the $\mathcal{O}(s^5)$ and $\mathcal{O}(s^4)$ sum rules, $\overline{\mathcal{M}}^{(3)}$ equals

$$\overline{\mathcal{M}}^{(3)} = \frac{5\,\kappa^2\,\sin^2\theta}{1152\,\pi r_c\, m_n^4}\left\{\sum_j \frac{m_j^4}{m_n^4}a^2_{nnj} - \frac{16}{15}a_{nnnn} - \frac{4}{5}\left[\frac{9\,b^2_{n'n'r}}{(m_n r_c)^4} - a^2_{nn0}\right]\right\}\ . \tag{5.77}$$

These contributions cancel if the following $\mathcal{O}(s^3)$ sum rule holds true:

$$\sum_{j=0}^{+\infty}\mu_j^4 a^2_{nnj} = \frac{16}{15}\mu_n^4 a_{nnnn} + \frac{4}{5}\left[9b^2_{n'n'r} - \mu_n^4 a^2_{nn0}\right] \tag{5.78}$$

We do not yet have an analytic proof of this sum rule; however we have verified that the right-hand side numerically approaches the left-hand side as the maximum intermediate KK number $N_{\max}$ is increased to 100 for a wide range of values of kr_c, including $kr_c \in \{10^{-3}, 10^{-2}, 10^{-1}, 1, 2, \ldots, 10\}$.[5]

The $\mathcal{O}(s^3)$ may also be written as

$$3\left[9b^2_{n'n'r} - \mu_n^4 a^2_{nn0}\right] = 15c_{nnnn} + \mu_n^4 a_{nnnn} \tag{5.79}$$

by applying Eq. (4.181) to Eq. (5.78).

---

[5] The cancellations implied by this sum rule can be seen in the vanishing of $\mathcal{R}^{[N](3)}$ Fig. 5.2 as N increases.

5.4.4 Cancellations at $\mathcal{O}(s^2)$ in RS1

The contributions to the elastic helicity-zero matrix element at $\mathcal{O}(s^2)$ equal

$$\overline{\mathcal{M}}^{(2)}_c = -\frac{\kappa^2\, a_{nnnn}}{54\,\pi r_c\, m_n^2}\left[5 + 47\cos(2\theta)\right] \ , \tag{5.80}$$

$$\overline{\mathcal{M}}^{(2)}_r = \frac{\kappa^2}{48\,\pi r_c\, m_n^2}\left[\frac{b^2_{n'n'r}}{(m_n r_c)^4}\right]\left[7 + \cos(2\theta)\right] \ , \tag{5.81}$$

$$\overline{\mathcal{M}}^{(2)}_0 = \frac{\kappa^2\, a^2_{nn0}}{576\,\pi r_c\, m_n^2}\left[175 + 624\cos(2\theta) + \cos(4\theta)\right] \ , \tag{5.82}$$

$$\begin{aligned}\overline{\mathcal{M}}^{(2)}_{j>0} = \frac{\kappa^2\, a^2_{nnj}}{6912\,\pi r_c\, m_n^2}\Bigg\{ & 4\left[7 + \cos(2\theta)\right]\left[5 - 2\frac{m_j^2}{m_n^2}\right]\frac{m_j^4}{m_n^4} \\ & - \left[1291 + 1132\cos(2\theta) + 9\cos(4\theta)\right]\frac{m_j^2}{m_n^2} \\ & + 4\left[553 + 1876\cos(2\theta) + 3\cos(4\theta)\right]\Bigg\} \ .\end{aligned} \tag{5.83}$$

By applying the $\mathcal{O}(s^5)$ and $\mathcal{O}(s^4)$ sum rules (but *not* the $\mathcal{O}(s^3)$ sum rule), the total $\mathcal{O}(s^2)$ contribution equals

$$\overline{\mathcal{M}}^{(2)} = \frac{\kappa^2\,\left[7 + \cos(2\theta)\right]}{864\,\pi r_c\, m_n^2}\left\{\sum_j\left[\frac{m_j^2}{m_n^2} - \frac{5}{2}\right]\frac{m_j^4}{m_n^4}a^2_{nnj} + \frac{8}{3}a_{nnnn} - 2\left[\frac{9\, b^2_{n'n'r}}{(m_n r_c)^4} - a^2_{nn0}\right]\right\} \ , \tag{5.84}$$

which vanishes if the following $\mathcal{O}(s^2)$ sum rule holds:

$$\sum_{j=0}^{+\infty}\left[\mu_j^2 - \frac{5}{2}\mu_n^2\right]\mu_j^4 a^2_{nnj} = -\frac{8}{3}\mu_n^6 a_{nnnn} + 2\mu_n^2\left[9b^2_{n'n'r} - \mu_n^4 a^2_{nn0}\right] \ . \tag{5.85}$$

Again, we do not yet have a proof for this sum rule, despite strong numerical evidence that it is correct (see Sec. 5.5). However, combining the $\mathcal{O}(s^3)$ and $\mathcal{O}(s^2)$ sum rules (Eqs. (5.78) and (5.85)), yields an equivalent set

$$\sum_{j=0}^{+\infty}\left[\mu_j^2 - 5\mu_n^2\right]\mu_j^4 a^2_{nnj} = -\frac{16}{3}\mu_n^6 a_{nnnn} \ , \tag{5.86}$$

$$3\left[9b^2_{n'n'r} - \mu_n^4 a^2_{nn0}\right] = 15 c_{nnnn} + \mu_n^4 a_{nnnn} \ . \tag{5.87}$$

where Eq. (5.86) is precisely Eq. (4.183) (which we proved in Section 4.4) and Eq. (5.87) is Eq.(5.79) again. Therefore, if the $\mathcal{O}(s^3)$ sum rule holds true, then the $\mathcal{O}(s^2)$ must also hold true, and vice versa. Of the relations necessary to ensure cancellations, only Eq. (5.87) remains unproven.

Finally, we note that the sum rules we have derived in RS1 in Eqs. (4.179), (4.180), (5.78), and (5.85), are consistent with those inferred by the authors of [32] who assumed that cancellations in the spin-0 scattering amplitude of massive spin-2 modes in KK theories must occur to result in amplitudes which grow like $\mathcal{O}(s)$. A description of the correspondence of our results with theirs is given in Appendix E of [18].

5.4.5 The Residual $\mathcal{O}(s)$ Amplitude in RS1

After applying all the sum rules above[6] (including Eq. (5.87), which lacks an analytic proof), the leading contribution to the elastic helicity-zero matrix element is found to be $\mathcal{O}(s)$. The relevant contributions, sorted by diagram type, equal

$$\overline{\mathcal{M}}^{(1)}_c = \frac{\kappa^2\, a_{nnnn}}{1728\,\pi r_c}\left[1505 + 3108\cos(2\theta) - 5\cos(4\theta)\right] \,, \tag{5.88}$$

$$\overline{\mathcal{M}}^{(1)}_r = -\frac{\kappa^2}{24\,\pi r_c}\left[\frac{b^2_{n'n'r}}{(m_n r_c)^4}\right]\left[9 + 7\cos(2\theta)\right] \,, \tag{5.89}$$

$$\overline{\mathcal{M}}^{(1)}_0 = \frac{\kappa^2\, a^2_{nn0}\,\csc^2\theta}{2304\,\pi r_c}\left[748 + 427\cos(2\theta) + 1132\cos(4\theta) - 3\cos(6\theta)\right] \,, \tag{5.90}$$

$$\begin{aligned}\overline{\mathcal{M}}^{(1)}_{j>0} = \frac{\kappa^2\, a^2_{nnj}\,\csc^2\theta}{6912\,\pi r_c}\Bigg\{ & 3\left[7+\cos(2\theta)\right]^2\frac{m_j^8}{m_n^8} - 4\left[241 + 148\cos(2\theta) - 5\cos(4\theta)\right]\frac{m_j^6}{m_n^6} \\ & + 4\left[787 + 604\cos(2\theta) - 47\cos(4\theta)\right]\frac{m_j^4}{m_n^4} \\ & - \left[3854 + 5267\cos(2\theta) + 98\cos(4\theta) - 3\cos(6\theta)\right]\frac{m_j^2}{m_n^2} \\ & + \left[2156 + 1313\cos(2\theta) + 3452\cos(4\theta) - 9\cos(6\theta)\right]\Bigg\} \,. \end{aligned} \tag{5.91}$$

Combining them, according to Eq. (5.36), yields

$$\overline{\mathcal{M}}^{(1)} = \frac{\kappa^2\,\left[7+\cos(2\theta)\right]^2\csc^2\theta}{2304\,\pi r_c}\left\{\sum_j \frac{m_j^8}{m_n^8}a^2_{nnj} + \frac{28}{15}a_{nnnn} - \frac{48}{5}\left[\frac{9\,b^2_{n'n'r}}{(m_n r_c)^4} - a^2_{nn0}\right]\right\} . \tag{5.92}$$

This is generically nonzero, and thus represents the true leading high-energy behavior of the elastic helicity-zero RS1 matrix element.

5.4.6 Other Helicity Combinations

The sum rules of the previous subsections were derived by considering what cancellations were necessary to ensure the elastic helicity-zero RS1 matrix element grew no faster than

[6] The elastic 5D Orbifolded Torus couplings (5.47) directly satisfy all of these sum rules.

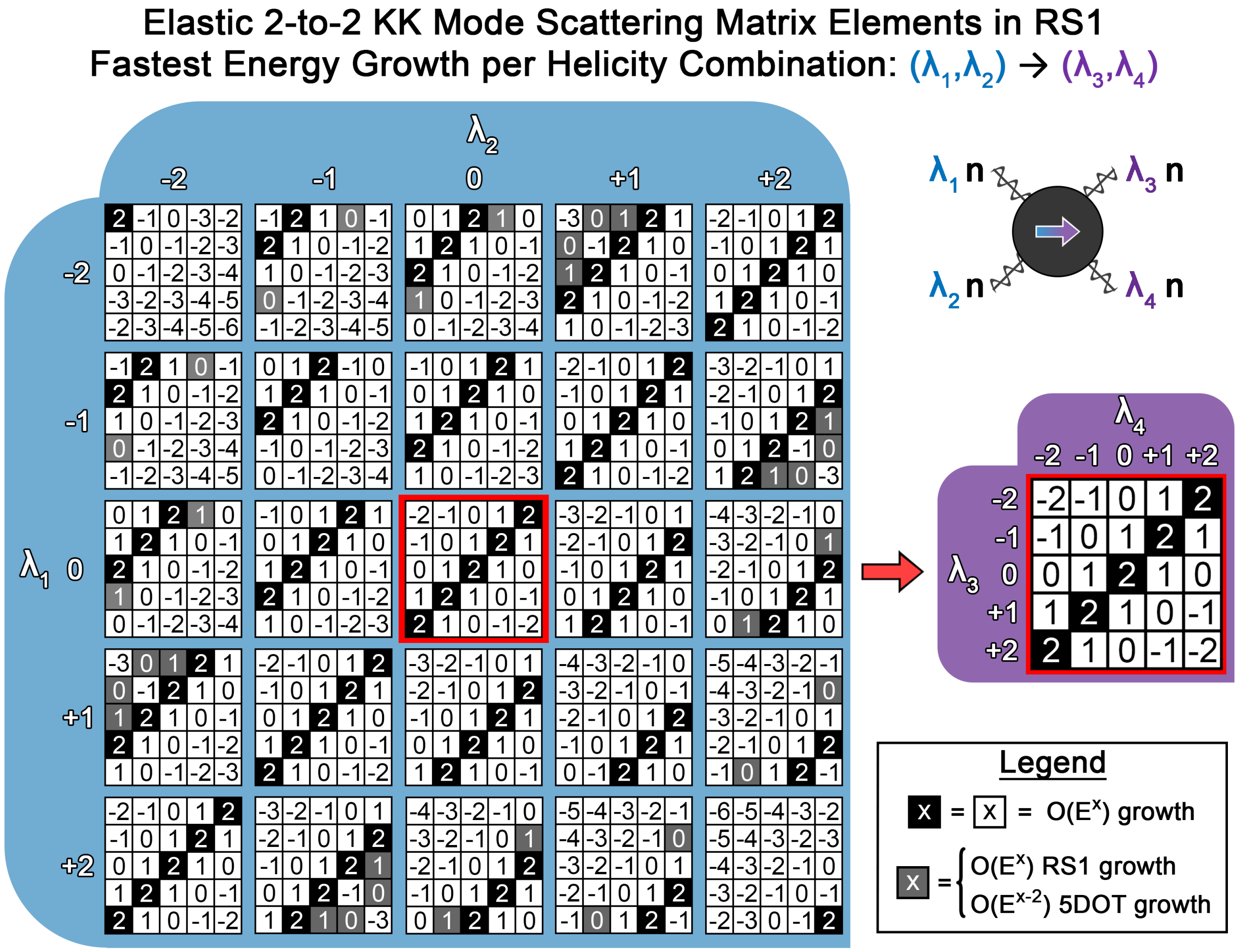


Figure 5.1: This table gives the leading order (in energy) growth of elastic $(n,n) \to (n,n)$ scattering for different incoming ($\lambda_{1,2}$) and outgoing ($\lambda_{3,4}$) helicity combinations in RS1. In the cases listed in grey, the leading order behavior is softer in the orbifolded torus limit (by two powers of center-of-mass energy).

$\mathcal{O}(s)$, a constraint which in turn comes from considering the extra-dimensional physics. This bound on high-energy growth must hold for scattering of all helicities.

Indeed, upon studying the nonzero-helicity scattering amplitudes, we find that the sum rules derived in the helicity-zero case are sufficient to ensure *all* elastic RS1 matrix elements grow at most like $\mathcal{O}(s)$.

Figure 5.1 lists the leading high-energy behavior of the elastic RS1 matrix element for each helicity combination after the sum rules have been applied. These results are expressed in terms of the leading exponent of incoming energy $E \equiv \sqrt{s}$. For example, the elastic helicity-zero matrix element diverges like $\mathcal{O}(s) = \mathcal{O}(E^2)$ and so its growth is recorded as "2" in the table. As expected, no elastic RS1 matrix element grows faster than $\mathcal{O}(E^2)$.

Some matrix elements grow more slowly with energy in the 5DOT model than they do in the more general RS1 model; they are indicated by the gray boxes in Fig. 5.1. For

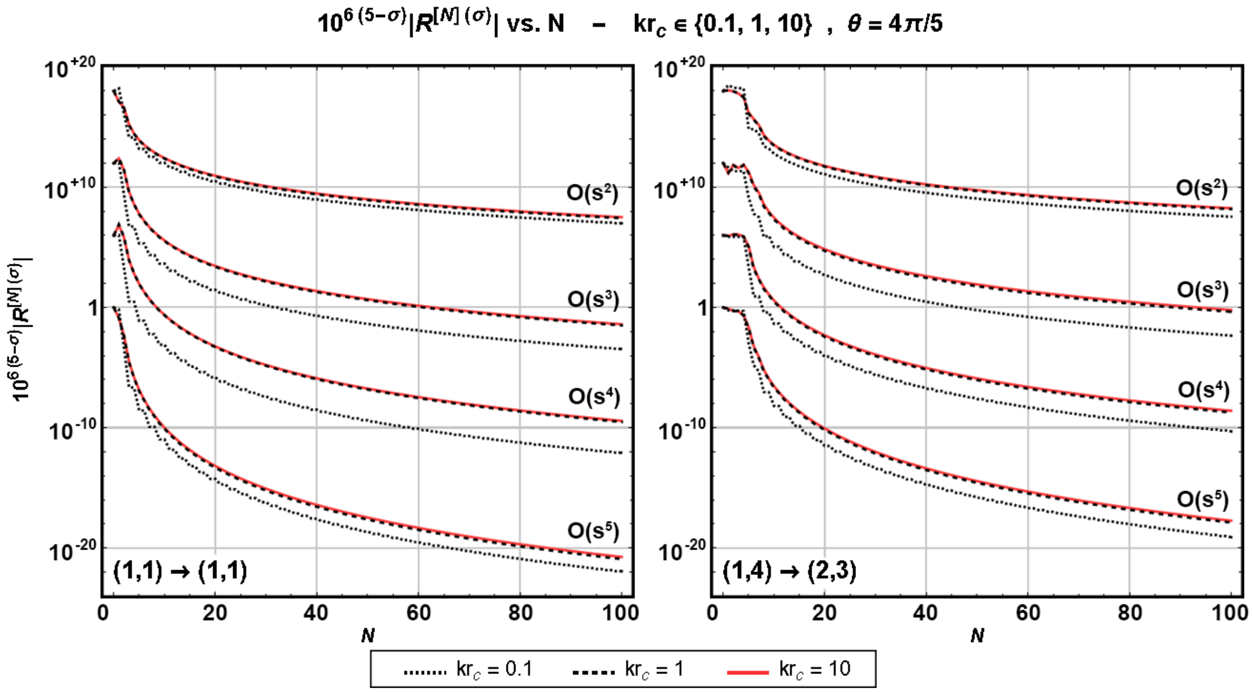


Figure 5.2: These plots show the ratio $\mathcal{R}^{[N](\sigma)}(kr_c, \theta) = \mathcal{M}^{[N](\sigma)}/\mathcal{M}^{[0](\sigma)}$ (defined in Eq. (5.97)), where $\mathcal{M}^{[N](\sigma)}$ is the $\mathcal{O}(s^\sigma)$ contribution to the matrix element describing helicity-zero scattering of KK modes $(1,1) \to (1,1)$ (left) and $(1,4) \to (2,3)$ (right) as a function of the number of KK intermediate states included in the calculation (N). The curves are shown for $kr_c = 0.1,\ 1,\ 10$ at fixed $\theta = 4\pi/5$. In all cases, the remaining matrix element falls rapidly with the addition of more intermediate states, thereby demonstrating the cancellation of all high-energy growth faster than $\mathcal{O}(s)$. To visually separate the different curves, the value of the ratio at $N = 0$ has been artificially normalized to $(1, 10^6, 10^{12}, 10^{18})$ for $\sigma = 5, 4, 3, 2$ respectively.

these instances, the leading $\mathcal{M}^{(\sigma)}$ contribution in RS1 is always proportional to the same combination of couplings

$$\left[3a_{nn0}^2 + 16a_{nnnn}\right]\mu_n^4 - 27\,b_{nnr}^2\ , \tag{5.93}$$

which vanishes exactly when kr_c vanishes. Regardless of the specific helicity combination considered, no full matrix element vanishes.

5.5 Numerical Study of Scattering Amplitudes in the Randall-Sundrum Model [7]

This section presents a detailed numerical analysis of the scattering in the RS1 model. In Sec. 5.5.1 we demonstrate that the cancellations demonstrated for elastic scattering occur for

[7] The content of this section was originally published as Section VI and Appendix F.3 of [18], up to minor changes in wording and notation.

inelastic scattering channels as well, with the cancellations becoming exact as the number of included intermediate KK modes increases. In Sec. 5.5.2 we examine the truncation error arising from keeping only a finite number of intermediate KK mode states. We then return, in Sec. 5.5.3 to the question of the validity of the KK mode EFT. In particular, we demonstrate directly from the scattering amplitudes that the cutoff scale is proportional to the RS1 emergent scale [12, 13]

$$\Lambda_\pi = M_{\text{Pl}}\, e^{-k\pi r_c} \ , \tag{5.94}$$

which is related to the location of the IR (TeV) brane [10, 11].

5.5.1 Numerical Analysis of Cancellations in Inelastic Scattering Amplitudes

We have demonstrated that the elastic scattering amplitudes in the Randall-Sundrum model grow only as $\mathcal{O}(s)$ at high energies, and have analytically derived the sum rules which enforce these cancellations. Physically, we expect similar cancellations and sum rules apply for arbitrary inelastic scattering amplitudes as well. However, we have found no analytic derivation of this property.[8]

Instead, we demonstrate here numerical checks with which we observe behavior consistent with the expected cancellations. To do so, we must first rewrite our expressions so we may vary kr_c while keeping M_{Pl} and m_1 fixed. We do so by noting that we may rewrite the common matrix element prefactor as

$$\frac{\kappa^2}{\pi r_c} = \frac{\kappa_{4\text{D}}^2}{\psi_0{}^2} = \frac{1}{\pi k r_c}\left[1 - e^{-2kr_c\pi}\right]\frac{4}{M_{\text{Pl}}^2} \ , \tag{5.95}$$

and that $r_c = \mu_1/m_1$, such that $\mathcal{M}^{(\sigma)}$ can be factorized for any process (and any helicity combination) into three unitless pieces, each of which depends on a different independent parameter:

$$\mathcal{M}^{(\sigma)} \equiv \left[\mathcal{K}^{(\sigma)}(kr_c,\theta)\right]\cdot\left[\frac{s}{M_{\text{Pl}}^2}\right]\cdot\left[\frac{\sqrt{s}}{m_1}\right]^{2(\sigma-1)} \ . \tag{5.96}$$

This defines the dimensionless quantity $\mathcal{K}^{(\sigma)}$ characterizing the residual growth of order $(\sqrt{s})^{2\sigma}$ in any scattering amplitude . We can apply this decomposition to the truncated matrix element contribution $\mathcal{M}^{[N](\sigma)}$, as defined in Eq. (5.37) as well. By comparing $\mathcal{M}^{[N](\sigma)}$ to $\mathcal{M}^{[0](\sigma)}$ and increasing N when $\sigma > 1$, we can measure how cancellations are improved by including more KK states in the calculation and do so in a way that depends only on kr_c and θ. Therefore, we define

$$\mathcal{R}^{[N](\sigma)}(kr_c,\theta) \equiv \frac{\mathcal{M}^{[N](\sigma)}}{\mathcal{M}^{[0](\sigma)}} = \frac{\mathcal{K}^{[N](\sigma)}}{\mathcal{K}^{[0](\sigma)}} \ , \tag{5.97}$$

[8] This is to be contrasted with the situation for KK compactifications on Ricci-flat manifolds, where an analytic demonstration of the needed cancellations has been found [32].

which vanishes as $N \to +\infty$ if and only if $\mathcal{M}^{[N](\sigma)}$ vanishes as $N \to +\infty$. Because $\mathcal{R}^{[N](\sigma)}$ depends continuously on θ, we expect that so long as we choose a θ-value such that $\mathcal{K}^{[N](\sigma)} \neq 0$, its exact value is unimportant to confirming cancellations. Figure 5.2 plots $10^{6(5-\sigma)}\mathcal{R}^{[N_{\max}](\sigma)}$ for the helicity-zero processes $(1,1) \to (1,1)$ and $(1,4) \to (2,3)$ as functions of $N_{\max} \to 100$ for $kr_c \in \{10^{-1}, 1, 10\}$ and $\theta = 4\pi/5$. The factor of $10^{6(5-\sigma)}$ only serves to vertically separate the curves for the reader's visual convenience; without this factor, the curves would all begin at $\mathcal{R}^{[0](\sigma)} = 1$ and thus would substantially overlap.

We find that, both for the case of elastic scattering $(1,1) \to (1,1)$ where we have an analytic demonstration of the cancellations and for the inelastic case $(1,4) \to (2,3)$ where we do not, $\mathcal{M}^{[N](\sigma)} \to 0$ as $N \to \infty$. Furthermore, we find that the rate of convergence is similar in the two cases. In addition, and perhaps more surprisingly, the rate of convergence is relatively independent of the value of kr_c for values between 1/10 and 10.

5.5.2 Truncation Error

In the RS1 model, the exact tree-level matrix element for any scattering amplitude requires summing over the entire tower of KK states. In practice, of course, any specific calculation will only include a finite number of intermediate states N. In this subsection we investigate the size of the "truncation error" of such a calculation. For simplicity, in this section we will focus on the helicity-zero elastic scattering amplitude $(1,1) \to (1,1)$ and investigate the size of the truncation error for different values of kr_c and center-of-mass scattering energy.

For $\sigma > 1$, consider the ratio

$$\mathcal{F}^{[N](\sigma)}(kr_c, s) \equiv \max_{\theta\in[0,\pi]} \left| \frac{\mathcal{M}^{[N](\sigma)}(kr_c, s, \theta)}{\mathcal{M}(kr_c, s, \theta)} \right| , \tag{5.98}$$

which measures the size of each truncated matrix element contribution relative to the full amplitude.[9] For sufficiently large N and $\sigma > 1$ we have confirmed numerically that the ratio $|\mathcal{M}^{[N](\sigma)}/\mathcal{M}^{[N}|$ reaches a global maximum at $\theta = \pi/2$ for $\sigma > 1$. Therefore

$$\mathcal{F}^{[N](\sigma)}(kr_c, s) = \left| \frac{\mathcal{M}^{[N](\sigma)}(kr_c, s, \theta)}{\mathcal{M}(kr_c, s, \theta)} \right|_{\theta=\pi/2} . \tag{5.99}$$

Unlike $\mathcal{M}^{(\sigma)}$ for $\sigma > 1$, $\mathcal{M}^{(1)}$ diverges at $\theta \in \{0, \pi\}$ because of a $\csc^2\theta$ factor, as indicated in Eq. (5.92), which arises from the t- and u-channel exchange of light states.[10] The total elastic RS1 amplitude $\mathcal{M}$, on the other hand, only has such IR divergences due to the exchange of the massless graviton and radion. For this reason, and as confirmed by the numerical evaluation of $\mathcal{M}^{[N](1)}/\mathcal{M}^{[N]}$, the divergences at $\theta \in \{0, \pi\}$ of $\mathcal{M}^{[N](1)}$ are actually slightly more severe than the corresponding divergences of $\mathcal{M}^{[N]}$, and so the ratio $\mathcal{M}^{[N](1)}/\mathcal{M}^{[N]}$ grows large in the vicinity of $\theta \in \{0, \pi\}$. However, this unphysical divergence

[9] In practice, we approximate the "full" amplitude by $\mathcal{M}^{[N=100]}(kr_c, s, \theta)$, which we have checked provides ample sufficient numerical accuracy for the quantities reported here.

[10] Formally, the sum over intermediate KK modes in Eq. (5.92) extends over all masses, but the couplings a_{11n} vanish as n grows and suppress the contributions from heavy states.

is confined to nearly forward or backward scattering; otherwise the ratio is approximately flat. Thus for $\sigma = 1$ we study the analogous quantity

$$\mathcal{F}^{[N](1)}(kr_c, s) = \left|\frac{\mathcal{M}^{[N](\sigma)}(kr_c, s, \theta)}{\mathcal{M}(kr_c, s, \theta)}\right|_{\theta=\pi/2} . \tag{5.100}$$

We also define the overall accuracy of the partial sum over intermediate states using a version of this quantity for which no expansion in powers of energy has been made:

$$\mathcal{F}^{[N]}(kr_c, s) \equiv \left|\frac{\mathcal{M}^{[N]}(kr_c, s, \frac{\pi}{2})}{\mathcal{M}(kr_c, s, \frac{\pi}{2})}\right| . \tag{5.101}$$

Because $\mathcal{F}^{[N](\sigma)}$ ($\mathcal{F}^{[N]}$) measures the discrepancy between any given contribution $\mathcal{M}^{[N](\sigma)}$ ($\mathcal{M}^{[N]}$) and the full matrix element $\mathcal{M}$, we study these quantities to understand the truncation error. In the upper two panes of Fig. 5.3 we plot the plot these quantities as a function of maximal KK number N for $kr_c = 1/10$ and $kr_c = 10$ at the representative energy $s = (10m_1)^2$, for $m_1 = 1$ TeV. The lower two panes of Fig. 5.3 plot similar information but at the energy $s = (100m_1)^2$. The $kr_c = 10$ panes contain the more phenomenologically relevant information. In all cases, we find that including sufficiently many modes in the KK tower yields an accurate result for angles away from the forward or backward scattering regime. When including only a small number of modes N, the contribution from $\mathcal{M}^{[N](5)}$ (the residual contribution arising from the noncancellation of the $\mathcal{O}(s^5)$ contributions) dominates and the truncation yields an inaccurate result. As one increases the number of included modes, this unphysical $\mathcal{O}(s^5)$ contribution to the amplitude falls in size until the full amplitude is dominated by $\mathcal{M}^{[N](1)}$, which is itself a good approximation to the complete tree-level amplitude. For $kr_c = 1/10$, the number of states N required to reach this "crossover", however, increases from 3 to 15 as $\sqrt{s}$ increases from $10m_1$ to $100m_1$. Consistent with our analysis in the previous subsection, however, the truncation error is less dependent on kr_c; the number of states required to reach crossover increases by less than a factor of 2 when moving from $kr_c = 1/10$ to $kr_c = 10$ at fixed $\sqrt{s}$.

Lastly, we note that the vanishing of $\mathcal{F}^{[N](3)}$ as N increases is a numerical test of the $\mathcal{O}(s^3)$ sum rule in Eq. (5.78).

5.5.3 The Strong-Coupling Scale at Large kr_c

In Section 5.3 we analyzed the tree-level scattering amplitude $(1,4) \to (2,3)$ and discovered that the 5D gravity compactified on a (flat) orbifolded torus becomes strongly coupled at roughly the Planck scale, $\Lambda^{(\text{5DOT})}_{\text{strong}} \equiv \sqrt{4\pi} M_{\text{Pl}}$. In the large kr_c limit of the RS1 model, however, we expect that all low-energy mass scales are determined by the emergent scale [12, 13]

$$\Lambda_\pi = M_{\text{Pl}}\, e^{-\pi k r_c} , \tag{5.102}$$

which is related to the location $\varphi = \pi$ of the IR brane [10, 11]. In this section we describe how this emergent scale arises from an analysis of the elastic KK scattering amplitude in the large-kr_c limit.

Consider the helicity-zero polarized $(n,n) \to (n,n)$ scattering amplitude. As plotted explicitly for $n=1$ in the previous subsection, at energies $s \gg m_n^2$ the scattering amplitude is dominated by the leading term $\mathcal{M}^{(1)}(kr_c, s, \theta)$ given in Eq. (5.92). The analogous expression in the 5D Orbifold Torus is given by Eq. (5.48). We note that the angular dependence of these two expressions is precisely the same, and therefore we can compare their amplitudes by taking their ratio. This gives the purely kr_c-dependent result[11]

$$\frac{\mathcal{M}^{(1)}(kr_c)}{\mathcal{M}^{(1)}(0)} = \left[\frac{1-e^{-2\pi kr_c}}{2\pi kr_c}\right] \cdot \overline{\mathcal{K}}_{nnnn}(kr_c) \ , \tag{5.103}$$

where

$$\overline{\mathcal{K}}_{nnnn} = \frac{1}{405}\left\{15\sum_j \frac{m_j^8}{m_n^8} a_{nnj}^2 + 28 a_{nnnn} - 144\left[\frac{9\, b_{nnr}^2}{(m_n r_c)^4} - a_{nn0}^2\right]\right\} . \tag{5.104}$$

From this ratio, we can estimate the strong-coupling scale at nonzero kr_c:

$$\begin{aligned} \Lambda^{(\text{RS1})}_{\text{strong}}(kr_c) &\equiv \Lambda^{(\text{RS1})}_{\text{strong}}(0)\sqrt{\frac{\mathcal{M}^{(1)}(0)}{\mathcal{M}^{(1)}(kr_c)}} \ , \\ &= \frac{\Lambda^{(\text{5DOT})}_{\text{strong}}}{\sqrt{\overline{\mathcal{K}}_{nnnn}}}\sqrt{\frac{2\pi kr_c}{1-e^{-2\pi kr_c}}} \ . \end{aligned} \tag{5.105}$$

where we can use our earlier $\Lambda^{(\text{5DOT})}_{\text{strong}} = \sqrt{4\pi} M_{\text{Pl}}$ result.

Now let us consider the kr_c dependence of this expression in the large kr_c limit. At large kr_c, Eq. (5.105) becomes

$$\Lambda^{(\text{RS1})}_{\text{strong}}(kr_c) \approx \sqrt{4\pi} M_{\text{Pl}} \sqrt{\frac{2\pi kr_c}{\overline{\mathcal{K}}_{nnnn}}} \ . \tag{5.106}$$

whereas, using Eqs. (4.102)-(4.109),

$$\frac{m_j^8}{m_n^8} a_{nnj}^2 \approx \frac{x_j^8}{x_n^8} C_{nnj}^2\,(kr_c)\, e^{2\pi kr_c} \ , \tag{5.107}$$

$$a_{nnnn} \approx C_{nnnn}\,(kr_c)\, e^{2\pi kr_c} \ , \tag{5.108}$$

$$\frac{b_{n'n'r}^2}{(m_n r_c)^4} \approx \frac{1}{x_n^4} C_{nnr}^2\,(kr_c)\, e^{2\pi kr_c} \ , \tag{5.109}$$

$$a_{nn0}^2 \approx C_{nn0}\,(kr_c) \ . \tag{5.110}$$

[11] Formally, as in the case of toroidal compactification, this amplitude has an IR divergence due to the exchange of the massless graviton and radion modes. By taking the ratio of the amplitudes in RS1 to that in the 5D Orbifolded Torus, the IR divergences cancel and we can relate the strong-coupling scale in RS to that in the case of toroidal compactification.

such that

$$\frac{\overline{\mathcal{K}}_{nnnn}}{2\pi kr_c} = \frac{e^{2\pi kr_c}}{810\pi\, x_n^8}\left\{15\sum_{j=1}^{+\infty} x_j^8\, C_{nnj}^2 + 28\, x_n^8\, C_{nnnn} - 1296\, x_n^4\, C_{nnr}^2\right\}\ , \tag{5.111}$$

In this expression, the $x_{j,n}$ are the jth and nth zeros of the Bessel function J_1, respectively; the constants C_{nnj}, C_{nnnn}, and C_{nnr} (defined explicitly in Subsection 4.3.5) are integrals depending only on the Bessel functions themselves. Therefore, focusing on the overall kr_c dependence, we find that

$$\Lambda^{(\mathrm{RS1})}_{\mathrm{strong}} \propto \sqrt{4\pi} M_{\mathrm{Pl}} e^{-\pi kr_c} = \sqrt{4\pi}\Lambda_\pi \tag{5.112}$$

at large kr_c, as anticipated. The precise value of the proportionality constant depends weakly on the process considered, and in the large-kr_c limit for the processes $(n,n) \to (n,n)$ we find

n	1	2	3	4	5
$\Lambda^{(\mathrm{RS1})}_{\mathrm{strong}}/\sqrt{4\pi}\Lambda_\pi$	2.701	2.793	2.812	2.819	2.822

. (5.113)

Since these results for the elastic scattering amplitudes follow from the form of the wave-functions in Eq. (4.92), similar results will follow for the inelastic amplitudes as well - and they will also be controlled by Λ_π.

We have also examined the dependence for lower values of kr_c via Eq. (5.105). We display the dependence of $\Lambda^{(\mathrm{RS1})}_{\mathrm{strong}}$ as a function of kr_c for the processes $(1,1) \to (1,1)$ and $(1,4) \to (2,3)$ in Fig. 5.4. In all cases, we find that the strong-coupling scale is roughly Λ_π.

Therefore, in the RS1 model, as conjectured under the AdS/CFT correspondence, all low-energy mass scales are controlled by the single emergent scale Λ_π.

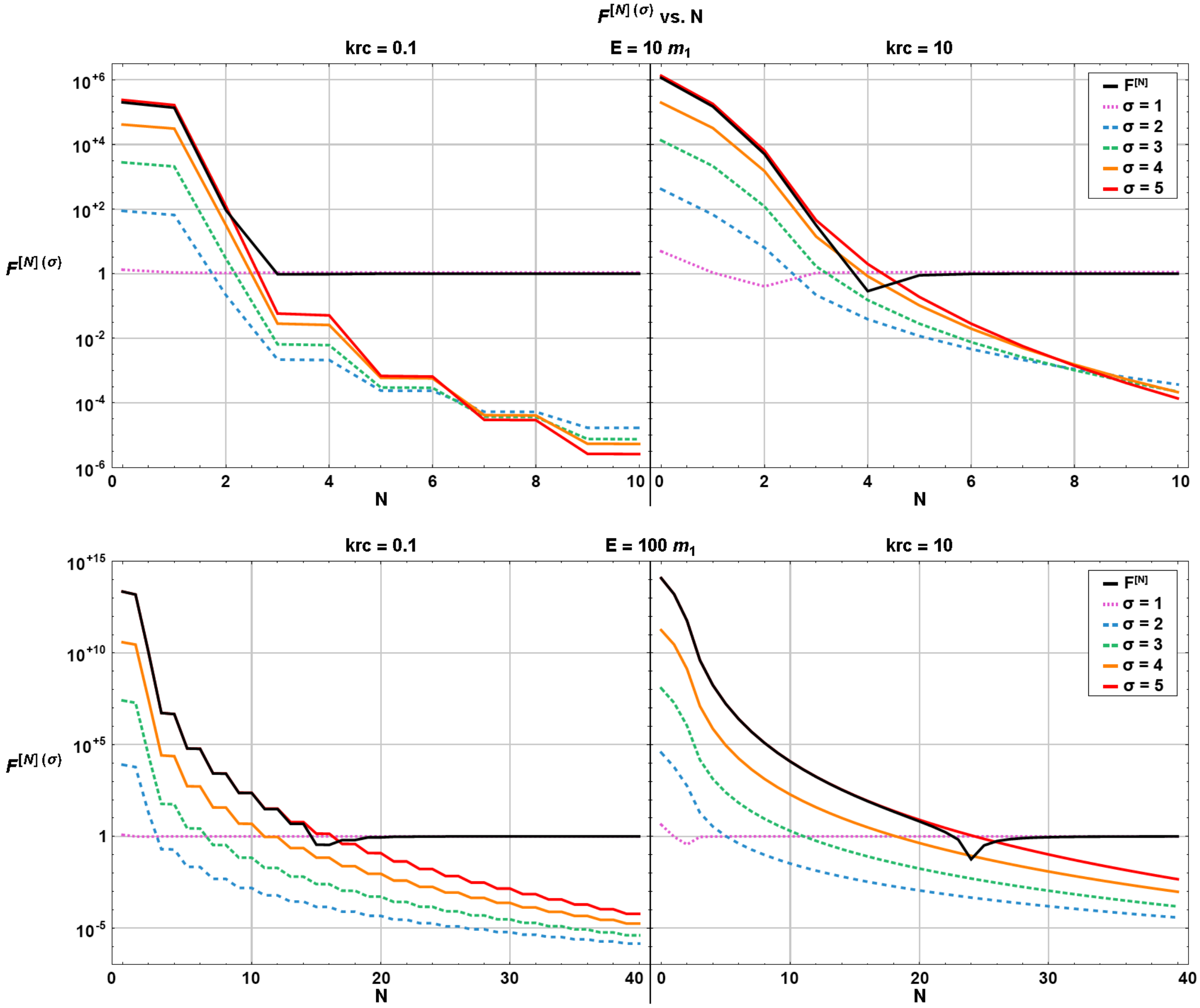


Figure 5.3: These plots show an upper bound on the size of the residual truncation error relative to the size of the full matrix element for the process $(1,1) \to (1,1)$ as a function of the number of included KK modes N, for $E = 10m_1$ (upper pair) and $E = 100m_1$ (lower pair), and $kr_c = 0.1$ (left pair) and $kr_c = 10$ (right pair). $\mathcal{F}^{[N](\sigma)}(kr_c, s)$ from Eq. (5.99) is shown in color, for $\sigma = 1$ - 5, and $\mathcal{F}^{[N]}(kr_c, s)$ from Eq. (5.101) is shown in black. We see that the size of the truncation error falls rapidly as the number of included intermediate states N increases. We also see that, for $E \gg m_1$, with a sufficient number of intermediate states $\mathcal{M}^{[N](1)}$ is a good approximation of the full matrix element. Note that if an insufficient number of intermediate KK modes is included, and the truncation error is large, $\mathcal{M}^{[N](5)}$ dominates.

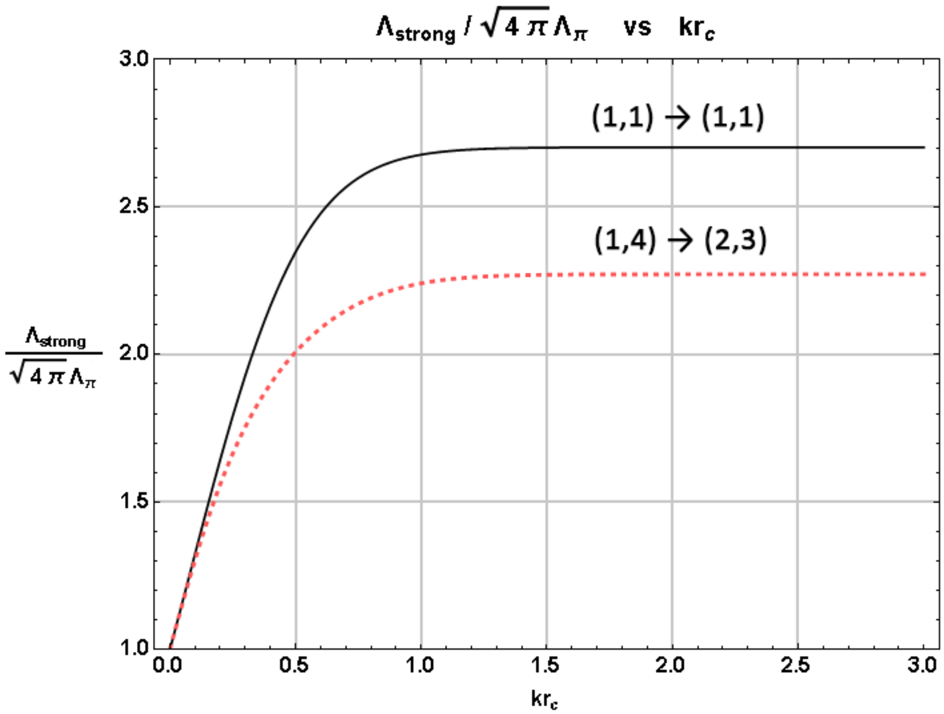


Figure 5.4: The strong-coupling scale $\Lambda^{(\mathrm{RS1})}_{\mathrm{strong}}(kr_c)$, Eq. (5.106), as a function of kr_c for the processes $(1,1) \to (1,1)$ and $(1,4) \to (2,3)$. We see that this scale is comparable to $\sqrt{4\pi}\Lambda_\pi$.

Chapter 6

Conclusion

Between what we published in [16, 17, 18] and additional original work discussed in this dissertation, we have obtained many substantial original results regarding the Randall-Sundrum 1 model:

- Summary of the 5D weak field expanded RS1 Lagrangian $\mathcal{L}_{\text{5D}}$ and its 4D effective equivalent $\mathcal{L}_{\text{4D}}^{(\text{eff})}$ through $\mathcal{O}(\kappa_{\text{5D}}^2)$. (Section 3.4 and Subsection 4.3.3.)

- Confirmation that all terms containing factors of $(\partial_\varphi|\varphi|)$ or $(\partial_\varphi^2|\varphi|)$ in $\mathcal{L}_{\text{5D}}$ are cancelled to all orders in the 5D coupling κ_{5D} in the full interacting theory. (Section 3.3.3)

- A new parameterization of the 4D effective RS1 Lagrangian as summarized in the 5D-to-4D formula, Eq. (4.65), which categorizes all couplings in the RS1 model as "A-type" or "B-type." (Section 4.3)

- The demonstration that the matrix element describing massive spin-2 KK mode scattering in the 5D orbifolded torus model yields $\mathcal{O}(s)$ growth for all helicity combinations. (Section 5.3)

- The demonstration that the matrix element describing massive spin-2 KK mode scattering in the RS1 model yields $\mathcal{O}(s)$ growth for all helicity combinations, including the derivation of sum rules that are sufficient for maintaining the cancellations from $\mathcal{O}(s^5)$ down to $\mathcal{O}(s)$. (Sections 5.4 and 5.5)

- Analytic proofs for many of the sum rules, as well as numerical evidence supporting the one rule lacking an analytic proof. (Section 4.4 and Figure 5.2)

- Numerical measurements of how KK tower truncation impacts the accuracy of the full matrix element and its $\mathcal{O}(s^\sigma)$ contributions ($\sigma \in \{1,2,3,4,5\}$) relative to the full matrix element without truncation. (Subsections 5.5.1 and 5.5.2)

- Calculation of the 5D strong-coupling scale $\Lambda_\pi = M_{\text{Pl}}\, e^{-kr_c\pi}$ directly from the 4D effective RS1 theory via partial wave unitarity constraints. (Subsection 5.5.3)

These results point toward several interesting open questions as well as providing a foundation for future work. There are several projects we will be pursuing (including some for which substantial progress has already been made):

- **The Role of the Radion:** The single sum rule which lacks an analytical proof is the combined $\mathcal{O}(s^3)$-$\mathcal{O}(s^2)$ rule, Eq. (5.79),

$$3\left[9b^2_{n'n'r} - \mu_n^4 a^2_{nn0}\right] = 15c_{nnnn} + \mu_n^4 a_{nnnn} \tag{6.1}$$

  and this owes to the curious coupling behavior of the radion. For example, the radion is introduced to the metric in the combination $\hat{u} \equiv (\kappa_{5\text{D}}\,\hat{r}/2\sqrt{6})\,\varepsilon^{+2}\,e^{-krc\pi}$, which means every instance of the 5D field $\hat{r}(x)$ carries with it a warp factor ε^{+2}, which throws a wrench in the otherwise powerful sum rules machinery developed in Section (4.4). Is an analytic proof of this sum rule possible? And if so, does it shed light on the role of the radion in the RS1 model?

- **Radion Stabilization:** The massless radion poses a problem for the RS1 model: if left as is, it generates an attractive Casimir force which pulls the branes at either end of the extra dimension together, thereby driving the extra dimension to smaller and smaller distance scales until the separation enters the quantum gravity regime and the RS1 model is no longer predictive [33, 34]. Furthermore, a massless radion would necessarily generate a scalar-tensor theory of long-distance gravitation at low energies contrary to the the usual pure tensor theory of 4D gravity. Therefore, phenomenological applications of the RS1 model require that the radion become massive in a process called radion stabilization. Radion stabilization typically involves adding a massive bulk scalar field to the RS1 Lagrangian that generates a radion potential which stabilizes the positions of the branes. However, we have found that adding a mass to the radion by hand causes the matrix elements describing massive spin-2 KK mode scattering to scale like $\mathcal{O}(s^2)$ instead of $\mathcal{O}(s)$. In a full model of radion stabilization, are cancellations down to $\mathcal{O}(s)$ maintained? If so, how does the introduction of radion stabilization influence the sum rules?

- **Bulk and Brane Matter:** Phenomenological applications of the RS1 model are not usually restricted to the pure gravity theory that we consider in this dissertation. Instead, physicists typically add either bulk or brane matter to the RS1 model, and investigate scattering of that matter in different circumstances. When adding (scalar, fermionic, vector) matter to the bulk or a brane, how do the new 2-to-2 scattering matrix elements scale at large energies? What new sum rules (if any) are implied?

  We have actually already completed the analyses of bulk and brane scalar matter, wherein we find that the process $\phi\phi \to h^{(n)}h^{(n)}$ for a bulk or brane scalar ϕ exhibits cancellations down to $\mathcal{O}(s)$—and derive several new sum rules.

- **Machinery:** Because of the complexity of diagrams involving multiple massive spin-2 particles, the analytic calculations required for the analyses in this dissertation were non-trivial. They required the development of a program that uses specialized techniques in order to complete the calculation in a timely fashion. It is our goal to generalize and clean up this code as to make it available for use to the wider physics community.

Thus, this dissertation presents original results about massive spin-2 KK mode scattering in the 4D effective Randall-Sundrum 1 model, and these results are of existing and future relevance in theoretical and phenomenological contexts.

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