RENORMALIZATION WITH THE GRADIENT FLOW

By

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PREFACE

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Part I

Exposition

This first part serves to provide a theoretical background of the forthcoming material. We begin with a brief summary of the origins and necessity of quantum field theory. This is followed by a more formal description upon which we construct quantum chromodynamics (QCD), the standard theory of the strong interaction governing the dynamics of quarks and gluons at high energies. QCD further gives rise to hadronic matter at intermediate scales and – residually, at low energies – to nuclear matter. Once the Lagrangian is constructed, we explore the perturbative expansion, gauge fixing, and renormalization. We discuss the impediments to performing calculations connecting disparate energy regimes, focusing in particular on the need for nonpeturbative methods and the concept of effective field theory (EFT); therein some degrees of freedom that are unresolvable at some reference scale are systematically removed, leaving a series of effective contributions characterizing the virtual presence of the full theory. This leads naturally into operator mixing and the operator-product expansion (OPE). These concepts are finally translated into the language of lattice field theory, giving us lattice quantum chromodynamics (LQCD), a numerical framework for studying QCD nonperturbatively on a discretized spacetime lattice. Since they are sensitive to the whole of the theory, lattice methods provide a natural setting for the study of bound states, such as the neutron, which are perturbatively inaccessible.

The second half of this chapter introduces baryon asymmetry and the Sakharov conditions for breaking such a symmetry, focusing on the violation of the combined charge-parity ($\mathcal{CP}$) symmetry. We discuss potential beyond-the-Standard-Model (BSM) sources of $\mathcal{CP}$ violation and their signatures in the EFT, leading to the concept of the neutron electric dipole moment (nEDM), a promising experimental probe of $\mathcal{CP}$ violation. This involves the insertion of effective operators into nucleonic correlation functions at hadronic energy scales accessible only nonperturbatively. We thus recast the problem into a lattice representation, where the QCD corrections to the effective sources may be determined numerically. Critically, this involves defining a renormalization scheme that is amenable to both the lattice and perturbation theory, while being sensitive to all operator mixing. For this we apply the gradient flow, the details of which are deferred to Part II.

The treatment here is moderately long, but the intention is that all techniques and formulae employed in the following parts be given a firm foundation and that this thesis should be reasonably self-contained. Important and lengthy calculations relating to this expository material are saved for the appendices.

Chapter 1

Quantum Field Theory

In the late 1920s, it became apparent that the quantum mechanics of Erwin Schrödinger and Werner Heisenberg could not fully treat the quantization of the electromagnetic field. Due to the manifest Lorentz covariance of Maxwell's equations, it was evident from the beginning that a properly quantized theory of electrodynamics should also exhibit this covariance. Unfortunately, the propagators of Schrödinger's theory were nonvanishing over the whole of spacetime, signaling a violation of causality. Moreover, Louis de Broglie's wavelike interpretation of the electron implied a wavelike nature for both the matter and forces in the quantum theory. To that end, Max Born, Pascual Jordan, and Heisenberg constructed a free field theory in 1925 by treating the degrees of freedom as an infinite set of quantized harmonic oscillators. Paul Dirac further showed in 1927 that this structure could replicate the Einstein coefficients. The pivotal step was, however, his introduction of the Dirac equation, the first successful relativistic wave equation. Schrödinger himself had first attempted to use the relativistic dispersion relation to construct his Hamiltonian, giving what would later be called Klein-Gordon equation for scalar fields. Lacking the full consideration of spin, this formalism could not reproduce the Bohr levels in hydrogen, so it was scrapped for the familiar Schrödinger equation.

As it turns out, it is not the Schrödinger equation, but the Hamiltonian operator that fails, perceived as acting on a single-particle Hilbert space. Indeed, this is partially [] why the original Klein-Gordon equation failed. Dirac, too, originally held a single-particle interpretation of his equation, implying for each state of energy E an accompanying state of energy $-E$. While immaterial in a free theory, the energy spectrum of the interaction Hamiltonian was unbounded below when including electrodynamics. Dirac proposed a sea of negative-energy eigenstates, all filled with negative-energy electrons save for a number of effectively positively-charged "holes," presumed to be protons. It was hoped that this would indirectly bound the Hamiltonian from below through the Pauli exclusion principle, but, notwithstanding a grave misinterpretation of the Fock space [], the stability of atoms and the vast discrepancy between the masses of the proton and electron were enough to condemn this picture []. Though the proton was out of the question, Dirac maintained that there was a fundamental importance to this symmetry under charge conjugation.

Carl Anderson's 1932 discovery of the positron rectified the situation. The "antielectron" field, formally identical to the electron bbut for its positive charge, replaced the Dirac sea; the positron not only fit the bill for the negative energy eigenstates of the Dirac equation, restoring the positive-definiteness of the Hamiltonian, but its oppositely-signed currents flowed backwards relative to those of the electron, fixing also the problem of propagation over spacelike intervals.

These developments form the basis of second quantization, wherein fields are promoted to local operators acting on a multiparticle Fock space of excitations of the vacuum. The field operators, themselves subjected to the canonical quantization conditions, compose a complete set of quantum harmonic oscillators with a particle corresponding to each excitation. The excitations are generated by the coefficients of the Fourier decomposition, having now been promoted to ladder operators acting on the multiparticle states. The coefficients of the positive-frequency terms locally produce particles, while their negative-frequency counterparts produce antiparticles. Much of this work on configuration-space is due to Wolfgang Pauli and Jordan, who proved the commutation relations were Lorentz invariant, and to Vladimir Fock, who constructed the Hilbert space and worked out — along with Eugene Wigner and Jordan — canonical (anti)commutation relations for bosons (fermions) consistent with spin and statistics []. All of these advances allowed for the construction of the S-matrix, which produces all observables.

In 1949, Freeman Dyson introduced the Dyson series, which gave a perturbative construction of the S-matrix. Specifically, his introduction of time-ordered correlation functions [] guaranteed causality by forcing amplitudes outside the lightcone to vanish. Gian Carlo Wick then proposed in 1950 a combinatorial decomposition of the matrix elements produced by the Dyson series. He related Dyson's time-ordered products to field contractions and normal-ordered products, the latter of which gave vanishing contributions to scattering amplitudes. This reduction expressed the matrix elements in terms of simple two-point functions and interaction vertices, forming from the local-field perspective a basis for Richard Feynman's diagrammatic approach.

Feynman himself preferred a particle theory, motivated by his and John Wheeler's earlier development of absorbers. They had built a generalized classical electrodynamics from the Lagrangian point of view, which was possible due to their unorthodox usage of both advanced and retarded waves and their dismissal of electrical self-interactions []. The interactions were confined to the lightcone with a delta distribution, which Feynman realized could be relaxed in a small neighborhood to generalize the behavior of electrodynamics at large energies. To explain the universality of the electron's mass and charge, Wheeler suggested that all electrons are one singular entity, traveling on a complicated, looping world line. Then on any time slice, those sections traveling toward the plane may be considered positrons, while those pointing away were electrons. Their action was particularly clean, and the absence of fields simplified both the mathematics and visualization. Feynman successfully incorporated fields into their theory, though he ultimately dismissed them as bikeshedding, leaving him suspicious of the Hamiltonian formalism. Indeed, when he moved on to developing a quantized absorber theory, he found his theory to be incompatible with the typical Hamiltonian methods of the time. He was later introduced to an idea of Dirac, that between two points in time, the path-dependence of a particle's trajectory could be related to an overall complex phase on the wavefunction, where the argument was proportional to the action along the path. Infinitesimally iterating along a finite interval, he found that the propagation of a particle could be described by a sum over all possible paths weighed by a phase equal to the associated action. This new path integral was a natural setting for his quantum absorber theory.

The measurement of the Lamb shift eventually forced Feynman to reconsider the self-energy of the electron. Following the suggestion of Hans Kramers, he worked with Hans

Bethe to calculate the self-energy in his path integral formalism, finding that the infinite result could be tamed by smearing the delta function in the Lagrangian for the absorber theory, amounting to a physical cutoff on the spacing of the points of self-interaction []. This was an early example of regularization and renormalization, where a measurable parameter is redefined to be the finite difference of two formally infinite quantities. Around this time, Feynman developed simpler methods for path integral calculations, culminating in his 1949 introduction of Feynman diagrams []. He, employing his heuristic "spacetime" method, proceeded to calculate the leading radiative correction to the electron's anomalous magnetic moment. Julian Schwinger and Shin'ichirō Tomonaga concomitantly arrived at the same result as Feynman through the local-field picture, to which Dyson subsequently proved the spacetime formulation was equivalent. The triplicate determination of the anomalous magnetic moment and Bethe's calculation of the Lamb shift were at this point the most accurate calculations in physics, bringing renormalization and the new quantum electrodynamics (QED) to the theoretical fore.

The 1950s and '60s were largely spent building models of the weak and strong interactions, catalyzed by Feynman's new, efficient methods and the continuing success of QED. There was a shift in attention to Noetherian symmetries and the role of Lie groups. Chen-Ning Yang and Robert Mills expanded the earlier work of Herman Weyl to describe the relationship between the allowable interactions and the symmetry group of the theory []. With the further work of Murray Gell-Mann providing physical consequences of group-theoretic considerations, quantum field theory matured into the study of gauge theories, the paradigm now being Yang-Mills theory. Additionally, the inclusion of fermions into the path integral was finally treated in full with the implementation of Grassmann calculus, owed chiefly to David Candlin [] (who is opprobriously absent from the modern literature). After Chien-Shiung Wu demonstrated a violation of parity in the electroweak interaction, there was a renewed interest in discrete symmetries as well, akin to the Dirac's earlier notion of charge symmetry. Sheldon Glashow, Adbus Salam, and John Ward developed a semisimple gauge theory unifying the weak force and electrodynamics, with Steven Weinberg supplying a mechanism for spontaneous symmetry breaking []. These successes inspired a Yang-Mills theory for Gell-Mann's quark model, quantum chromodynamics, but the confinement of quarks would not receive a satisfactory treatment until the demonstration by David Gross and Frank Wilczek that asymptotic freedom could be dynamically realized from the self-interaction of the gauge field. All of this was enabled by two major advancements. The first was the general gauge-fixing procedure of Ludvig Fadeev and Victor Popov, which removed the overcounting of gauge configurations in the path integral []. The second was a deeper understanding of renormalization granted by both the proof [] by Gerardus 't Hooft and Martinus Veltman that Yang-Mills theories are renormalizable and the identification of the renormalization group by Kenneth Wilson.

The '70s saw the grand synthesis of quantum and statistical field theories, led by Wilson's systematization of the scaling principles of Curtis Callan and Kurt Symanzik []. Wilson viewed the cutoffs of conventional renormalization as threshold scales beyond which the laws of physics were unknown. He explained how to encode irresolvable high-energy phenomena at low energies by successively "integrating out" highly energetic degrees of freedom, thus recasting current theories as effective theories for an as-yet-unknown ultraviolet (UV) theory. In this way, divergences induced by highly local and energetic interactions were seen to be artifacts of sending Wilson's thresholds to infinity. Renormalization was then just a demand that the physics must be insensitive to the mathematical choices made in imposing a cutoff. In an attempt to probe the nonperturbative confinement of quarks in hadrons, Wilson proposed defining the theory on a discrete spacetime lattice from which the physical theory could be recovered in the continuum and infinite volume limits []. Path integrals constituted an especially natural setting for this lattice field theory, and the path integral formula for scattering amplitudes could be easily translated to the discrete language. While perturbation theory had been extremely successful for weak interaction strengths, one could now generate numerical predictions for strongly coupled theories. A critical component of lattice field theory is the Euclideanization of the action. Interestingly, this stipulation led to the only mathematically rigorous definition of the path integral.

Anticipating the discovery of the Higgs boson in 2012 and the confirmation of the Glashow-Weinberg-Salam model for electroweak unification, the superstructure of the Standard Model (SM) of particle physics was reasonably complete, standing as the most precise and predictive theory of Nature ever constructed. A major blemish on its record, however, has been its inability to account for the obvious asymmetry of matter and antimatter in the Universe. Aside from an exceedingly small contribution from the electroweak sector, the SM predicts a largely democratic universe, producing matter and antimatter at roughly equal rates. Much of the work following Wilson was dedicated to unification of the forces and beyond-the-Standard-Model (BSM) extensions to fill in the proliferating gaps between theory and experiment. A standard feature of these theories is a measurable violation of the discrete charge-parity (CP) symmetry, in concordance with the Sakharov conditions for baryogenesis. The mechanisms for CP-violation are typically mediated by heavy particles, detectable only at very large energies. It is conceivable, however, that signatures of this broken symmetry are visible in very accessible systems, the archetypical example being the hypothetical neutron electric dipole moment (nEDM). Due to confinement at low energies, baryons are in general poorly defined in perturbation theory. On the other hand, Wilson's lattice theory is perfectly suited for these low-energy systems. In this regime, the high-energy BSM interactions are irresolvable. Instead, one may consider effective local interactions built from only the low-energy modes of the theory. In the Wilsonian picture, the low-energy Lagrangian is supplemented with an infinite tower of effective operators, corresponding to the potential UV completions. The potential contribution of each such interaction to the nEDM may be computed on the lattice by inserting the operators into hadronic matrix elements with electromagnetic currents. After a suitable renormalization, these results can be compared with several experimental measurements to isolate the physical contributions and identify appropriate BSM extensions. As we will discuss in Ch. 4, the renormalization of these operators is highly nontrivial on the lattice, forming the motivation for current manuscript. In what follows, we develop a method for circumventing the difficulties associated with lattice renormalization.

1.1 Relativistic Field Theory

The principal difference between quantum field theory and quantum mechanics is Lorentz covariance. Since the action is relativistically invariant, the Lagrangian formulation of classical field theory provides a natural foundation for a relativistic theory. The specific approach we take in constructing a QFT relies on the Feynman path integral, which expresses a quantum field theory with the manifest symmetries of the Lagrangian perspective. More importantly, when the strength of interaction for some phenomenon is too large, typical perturbative approximation techniques become invalid. The only systematic nonperturbative treatment of a quantum field theory is lattice field theory, which relies wholly on the discretization of the path integral. When the theory is QCD, the discrete analogue is lattice quantum chromodynamics (LQCD, Sec. 3), which is central to the following chapters.

The fundamental object containing the entire dynamics of a field theory is the Lagrangian density functional:

$$\mathcal{L} = \mathcal{L}[\{\phi_i\}, \{\dot{\phi}_i\}](x), \tag{1.1}$$

where ϕ_i and $\dot{\phi}_i$, $i \in [n]$ represent some n fields and their conjugate momenta. These degrees of freedom assume the traditional role of generalized coordinates, and since they are themselves functions of both spacial and temporal coordinates, the action is defined over the whole of spacetime:

$$S[\{\phi_i\}, \{\dot{\phi}\}] = \int d^4x \mathcal{L}[\{\phi_i\}, \{\dot{\phi}_i\}](x). \tag{1.2}$$

For each species of dynamical field, the Lagrangian contains a free-field contribution determined by its spin, for example

spin-0, real scalar ϕ	Klein-Gordon Lagrangian	$\mathcal{L}_{KG} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2}\mu^2\phi^2,$
spin-1/2, spinors $\bar{\psi}, \psi$	Dirac Lagrangian	$\mathcal{L}_D = \bar{\psi}(i\not{\partial} - m)\psi,$
spin-1, vector A_μ	Proca Lagrangian	$\mathcal{L}_P = \frac{1}{g^2} \operatorname{Tr}\left[\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - M^2 A_\mu A^\mu\right],$

and so on. The interactions are then determined by the imposition of a local gauge symmetry, following the Yang-Mills construction, which is generated by the unitary action a compact (semi)simple Lie group G at each spacetime coordinate. In order that the theory remain invariant under the action of such a group, the ordinary derivative must be promoted to a gauge covariant derivative,

$$\partial \to D = \partial + A. \tag{1.3}$$

The second term is the gauge connection, which is associated with the gauge boson generated by the symmetry group and takes values in the Lie algebra. Because derivatives are formally defined at two infinitesimally separated points, the local transformation acts on the field at each point separately. A tracks the change in the gauge transformation between these points. In particular, its transformation under the gauge group exactly cancels the change in the transformation between the two points under the derivative, ensuring local invariance of the action. To illustrate, consider a local gauge transformation $g \in G$ parametrized by the exponential map,

$$g(x) = e^{-\omega(x)}, \quad \omega(x) = \omega^a(x)t^a, \tag{1.4}$$

where t^a are the generators of the Lie algebra $\mathfrak{g}$, that acts on fermions as

$$\psi \xrightarrow{g} \psi' = e^{-\omega}\psi, \quad \bar{\psi} \xrightarrow{g} \bar{\psi}' = \bar{\psi}e^{-\omega^\dagger} = \bar{\psi}e^{\omega}. \tag{1.5}$$

Under the action of g, the kinetic term in the Dirac Lagrangian differentiates both the gauge transformation and the fermion field, so that

$$\mathcal{L}_D \xrightarrow{g} \mathcal{L}'_D = \bar{\psi} g^\dagger [i(\partial\!\!\!/ g) + ig\partial\!\!\!/ - mg]\psi = \mathcal{L}_D - i\bar{\psi}(\partial\!\!\!/ \omega)\psi. \tag{1.6}$$

If we consider the covariant derivative, the connection transforms as

$$A \xrightarrow{g} A' = gAg^\dagger + g\partial g^\dagger, \tag{1.7}$$

since it is in the adjoint representation. Hence

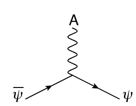
$$i\bar{\psi} A\!\!\!/ \psi \xrightarrow{g} i\bar{\psi}' A\!\!\!/' \psi' = i\bar{\psi} g^\dagger g(A\!\!\!/ - \partial\!\!\!/ \omega^\dagger) g^\dagger g\psi = i\bar{\psi} A\!\!\!/ \psi + i\bar{\psi}(\partial\!\!\!/ \omega)\psi, \tag{1.8}$$

which precisely compensates for the new term in Eq. 1.6.

In general, there will be a new term in D associated to each simple factor of a semisimple gauge group, since the Lie algebra is then a direct sum of simple algebras. In this way, each factor of the gauge group generates a gauge boson. Additionally, since the symmetry is continuous by construction, it corresponds to a conserved charge by Noether's theorem. More specifically, there is a conserved charge for each generator of the Lie algebra. We will treat this construction specifically in the case of quantum chromodynamics, Sec. **??**.

When inserted into the Dirac Lagrangian, the connection term in the covariant derivative produces an interaction of the form

$$i\bar{\psi} A\!\!\!/ \psi = \quad , \tag{1.9}$$



which characterizes the radiation of an A boson by an initial fermion $\bar{\psi}$ that exits in the state ψ. This specific interaction, analogous to the canonical momentum of classical electrodynamics, is called minimal. Of course, there may be other gauge-invariant interactions, but our later discussion of the renormalization group (Sec. 1.9) will clarify their absence from typical applications.

1.2 The Generating Functional

Once we have developed the action, we may encode the theory into a sum over histories, the generating functional;

$$Z = \int \mathcal{D}\phi \; e^{iS[\phi]}, \tag{1.10}$$

where the integral is meant to be taken over all configurations of the fields, ϕ. The generating functional acts as a partition function for the field configurations, where states are distributed according to the "Boltzmann" factor e^{iS}. The addition of static source fields to the Lagrangian permits the use of the Schwinger-Dyson equations to generate all Green's functions, or expectation values. Specifically, for some dynamical field ϕ and static source J, the Lagrangian is augmented by

$$\mathcal{L}_{source} = J\phi. \tag{1.11}$$

Successive functional differentiation of the path integral with respect to the source at some coordinates x_i brings down as many powers of $i\phi(x_i)$. Shutting off the sources and normalizing by the source-free generating functional $Z_0 = Z[J = 0]$ to remove an infinite background of vacuum fluctuations, we are able to compute all physical observables. Given some operator

$$\mathcal{O}(x_1, \ldots, x_N) = \Gamma\phi(x_1)\cdots\phi(x_N), \tag{1.12}$$

where the differential, spacetime, and gauge structures are generically encoded in the quantity Γ, we have

$$\langle\mathcal{O}\rangle = \frac{1}{Z_0}\int \mathcal{D}\phi\ \mathcal{O}e^{iS[\phi]} = \prod_{i=1}^{N}\frac{-i\delta}{\delta J(x_i)}\frac{1}{Z_0}Z[J]\bigg|_{J=0}. \tag{1.13}$$

This formula elucidates the probabilistic nature of the path integral; it gives the expectation value for a function $\mathcal{O}$ of random variable ϕ distributed by e^{iS}. With the knowledge of all correlation functions, the theory is effectively solved, although this is generally easier stated than practiced. Eq. 1.13 indeed generates all interactions, but we must carefully unpack it before defining both its perturbative and nonperturbative treatments.

1.3 Grassmann Numbers

First we discuss the implementation of fermions. In order to uphold Fermi-Dirac statistics, any two spinor fields must anticommute. This is accomplished by treating fermions as Grassmann numbers, which are most easily characterized algebraically. Indeed, there needn't be a a rigorous justification for the entire calculus, since ultimately we will only be interested in integration over entire factors of the Grassmann algebra. Let $\{\theta_i\}$ for $i \in [n]$ be the generators of an 2^n-dimensional unital algebra (the Grassmann algebra) over the complex numbers with the multiplicative law

$$\theta_i\theta_j + \theta_j\theta_i = \{\theta_i, \theta_j\} = 0, \quad \forall i, j \in [n]. \tag{1.14}$$

A direct consequence is that every generator is a zero divisor, in particular a square root of zero:

$$\theta_i^2 = \frac{1}{2}\{\theta_i, \theta_i\} = 0. \tag{1.15}$$

It follows that Taylor series truncate quickly; all functions are at most affine:

$$f(\theta) = a + b\theta, \tag{1.16}$$

for some complex a, b. We can thus define the Berezin integral of a function of a single Grassmann number []:

$$\int d\theta f(\theta) = a\int d\theta + b\int d\theta\ \theta, \tag{1.17}$$

where we have assumed linearity over the complex numbers. With the additional requirement that, since we are integrating over all θ, the integral must be translationally invariant, we have

$$\int d\theta\ f(\theta) = a\int d\theta + b\int d\theta\ (\theta + \eta) = (a - b\eta)\int d\theta + b\int d\theta\ \theta, \tag{1.18}$$

for another Grassmann number η. The first integral must vanish, since

$$a \int d\theta = (a - b\eta) \int d\theta, \tag{1.19}$$

and a, b are generic. The second integral above is simply an arbitrary normalization factor, with the conventional choice of

$$\int d\theta \; \theta = 1. \tag{1.20}$$

Multiple integration is easily found by extension, with the convention that for n variables θ_i,

$$\int d\theta_n \cdots \int d\theta_1 \; \theta_1 \cdots \theta_n = 1. \tag{1.21}$$

We are now able to calculate multivariate Gaussian integrals,

$$\int d\bar{\theta}_1 d\theta_1 \cdots \int d\bar{\theta}_n d\theta_n \; e^{-\bar{\theta}_i A_{ij} \theta_j},$$

for some $2n$ generators $\theta_i, \bar{\theta}_i$ and an n-dimensional Hermitian matrix A. Since the only terms that survive the Taylor expansion are linear in each variable, and each of these is totally antisymmetric, a unitary rotation U of the variables contributes an overall factor of $\det U$. Thus, by diagonalizing A we find the Matthews-Salam formula [];

$$\int d\bar{\theta}_1 d\theta_1 \cdots \int d\bar{\theta}_n d\theta_n \; e^{-\bar{\theta}_i A_{ij} \theta_j} = \det A, \tag{1.22}$$

contrasting the standard Gaussian integral which goes as $1/\sqrt{\det A}$. We may now express the Dirac field as a linear combination of Grassmann numbers ψ_i with coefficients $u_i(x)$ forming a basis for Dirac spinors:

$$\psi(x) = u_i(x) \psi_i, \tag{1.23}$$

where the Einstein summation convention is implied; we will adopt this notation for the rest of this work, unless otherwise specified.

1.4 Perturbation Theory

The full generating functional is rarely exactly soluble []. The free field theory, on the other hand, consists exclusively of quadratic actions and can be transformed into a product of manifestly integrable Gaussians. Denoting by g a generic coupling generated by a gauge interaction, the Lagrangian of any theory may be decomposed into free (0) and interacting (I) pieces (and perhaps a source term (S)),

$$\mathcal{L} = \mathcal{L}_0 + g\mathcal{L}_I \qquad (+\mathcal{L}_S), \tag{1.24}$$

where exclusively the interaction Lagrangian may contain higher powers in the coupling. For some fields collectively referred to as ϕ, we see immediately that

$$\langle \mathcal{O} \rangle = Z^{-1}[0] \int \mathcal{D}\phi \; e^{i \int \mathcal{L}} \mathcal{O} = Z^{-1}[0] \int \mathcal{D}\phi \; e^{i \int \mathcal{L}_0} e^{i \int \mathcal{L}_I} \mathcal{O} = \frac{Z_0[0]}{Z[0]} \langle e^{i \int \mathcal{L}_I} \mathcal{O} \rangle_0, \tag{1.25}$$

where we have absorbed the source term into $\mathcal{L}_0$ with a suitable shift of variables and defined the free partition function implicitly:

$$\langle \mathcal{O} \rangle_0 = Z_0^{-1}[0] \int \mathcal{D}\phi \; e^{i \int \mathcal{L}_0} \mathcal{O}, \tag{1.26}$$

with the obvious normalization:

$$Z_0[0] = \int \mathcal{D}\phi \; e^{i \int \mathcal{L}_0}, \tag{1.27}$$

which gives us

$$Z[0] = \int \mathcal{D}\phi \; e^{i \int \mathcal{L}} = Z_0[0] \left\langle e^{i \int \mathcal{L}_I} \right\rangle. \tag{1.28}$$

Thus,

$$\langle \mathcal{O} \rangle = \frac{\langle e^{i \int \mathcal{L}_I} \mathcal{O} \rangle_0}{\langle e^{i \int \mathcal{L}_I} \rangle_0}, \tag{1.29}$$

is now a partition function over the distribution defined by $\mathcal{L}_0$. There is a subtlety here. As written the path integral does not converge, because the Boltzmann factor is oscillatory and not positive definite. We may for now regulate this integral by adding a infinitesimal imaginary shift to the mass of each field, or equivalently adding to the Lagrangian a term

$$\mathcal{L}_\epsilon = i\epsilon \int \phi^2 \tag{1.30}$$

for each field (or pair of adjoint fields) ϕ and defining expectation values in the limit as $\epsilon \to 0$:

$$\langle \mathcal{O} \rangle = \lim_{\epsilon \to 0} Z^{-1}[0] \int \mathcal{D}\phi \; e^{i \int \mathcal{L} + i \int \mathcal{L}_\epsilon} \mathcal{O}. \tag{1.31}$$

We will often ignore this entirely, keeping the factor of $i\epsilon$ completely implicit. Indeed, when we pass to Euclidean space in Ch. 2 the limit commutes with the integral, and we may remove it explicitly. Now that we have a well-behaved generating functional, the form 1.29 begs a formal series expansion in g:

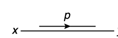
$$\langle e^{i \int \mathcal{L}_I} \mathcal{O} \rangle_0 = \sum_{i=0}^{\infty} \frac{i^n}{n!} \cdot g^n \left\langle \left(\int \mathcal{L}_I \right)^n \mathcal{O} \right\rangle_0. \tag{1.32}$$

Since the distribution here is a (regulated) multivariate Gaussian, we are free to invoke Isserlis' theorem:

$$\langle \phi_1 \cdots \phi_{2n} \rangle_0 = \frac{1}{2^n n!} \sum_{\pi \in S_{2n}} s_\pi \langle \phi_{\pi(1)} \phi_{\pi(2)} \rangle_0 \cdots \langle \phi_{\pi(2n-1)} \phi_{\pi(2n)} \rangle_0, \tag{1.33}$$

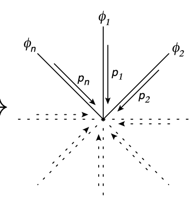
where by S_{2n} we denote the symmetric group on $2n$ letters, and $s_\pi = \pm 1$ represents a symmetrization factor for potentially anticommuting fields. Note that we chose an even number of fields so that the correlator does not trivially vanish. The denominator of Eq. 1.25 is clearly the vacuum expectation value, which contributes only vacuum fluctuations without external states. Typically then, we will simply ignore this normalization and define the generating functional to produce only amplitudes with external fields. At each order in Eq. 1.32, the integrand is a sum over all allowed contractions of the fields. Each set of contractions is decomposed into a number of free propagators and some vertex factors coming from $\mathcal{L}_I$ with an integral for each vertex that preserves the locality of the interaction. The propagators for each species of particle are simply the Green's functions for its equations of motion. Vertex factors are determined by taking the n-point functions at leading order and removing the external propagators. Together, the sets of values assigned to the propagators and fundamental vertices form the Feynman rules for the theory, which allow amplitudes to be built pictorially and calculated expediently. It is typically simplest to define them in momentum space, where the mathematical expressions are fairly uncomplicated. Roughly, propagators of momentum p are represented by oriented lines,

$$\langle \phi(x_1)\phi(x_2)\rangle \xrightarrow{\mathcal{F}} (2\pi)^4\delta^{(4)}(p_1+p_2)\langle\tilde{\phi}(p_2)\tilde{\phi}(p_1)\rangle \to \tag{1.34}$$



while vertices are given by an intersection of a number rays equal to the number of interacting fields:

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle \xrightarrow{\mathcal{F}} (2\pi)^4\delta^{(4)}(p_1+\cdots+p_n)\langle\tilde{\phi}(p_1)\cdots\tilde{\phi}(p_n)\rangle \to \tag{1.35}$$



With some more rules ensuring the proper weight of each contribution, each term in Eq.1.33 may be written as a Feynman diagram, which is labeled according to the rules above. Feynman diagrams will be the primary tool for calculations in this thesis. In Appendices B and C, we calculate the Feynman rules for Euclidean QCD explicitly and detail methods of calculating expectation values through the use of Feynman diagrams.

In a perturbative series, closed loops appear when there are fewer external fields than interacting fields at any order in the coupling. Each increasing order has an extra factor of the interaction Lagrangian, which increases the number of loops. We then speak of these interchangeably; the first nonvanishing order is defined to be the zeroth order or the "tree level," while subsequent orders n are called n-loop corrections. Explicitly, if we describe some function f as a series in the coupling g beginning at order g^m, then we have

$$f = g^m \sum_{n=0}^{\infty} f^{(n)} g^n. \tag{1.36}$$

In this case $f^{(0)}$ is the tree-level contribution, and the other $f^{(n)}$ are n-loop corrections. This should clarify our later discussion of vacuum diagrams, which have no external states and therefore contain loops in their tree-level contributions.

1.5 Gauge Fixing

Suppose we want to calculate the propagation amplitude for a free vector boson A between two points x and y, that is, the Green's function for the Yang-Mills equations of motion with zero coupling:

$$\langle A_\nu(y)A_\mu(x)\rangle = \int \mathcal{D}A\ A_\nu(y)A_\mu(x)e^{iS[A]}. \tag{1.37}$$

Since we are working in the free theory, where all interactions are quadratic, this integral should be exactly solvable by Gaussian integration (*v.i.*, App. B). Transforming to momentum space with four-momentum q, the Lagrangian goes as

$$\mathcal{L} \sim \int A^\nu(q^2 g_{\mu\nu} - q_\mu q_\nu)A^\mu, \tag{1.38}$$

where the expression in parentheses has a null eigenvector q_μ, corresponding to the unphysical longitudinal polarization of A, making it singular. In order to extract a Green's function, we must somehow remove this unphysical degree of freedom and fix the gauge. Unfortunately, removing removing the unphysical degrees of freedom destroys the gauge invariance and the unitarity of the path integral.

Faddeev and Popov solved this problem for generally nonabelian gauge theories by removing redundant gauge configurations in the functional integral []. Notice that the gauge invariance of the action partitions the symmetry group G into equivalence classes consisting of the (infinite) orbit of each configuration A. Each orbit contributes an infinite volume factor to the functional integral which represents the infinite number of physically indistinct configurations within it. Consequently, the integral measure overcounts each gauge orbit and is not normalizable by any finite volume. By restricting to a surface which intersects each gauge orbit once, we may restrict the domain of integration to the set of representatives of each orbit; in other words, we fix the gauge by imposing a constraint on the action in the form of a functional F such that

$$F[A] = 0. \tag{1.39}$$

Guaranteeing that the induced surface intersects each gauge orbit once is, however, impossible for the entire space of configurations, since a global section (global basis of coordinates) cannot be defined for nonabelian theories in general. This characterizes the Gribov ambiguity in choosing a representative for each orbit with a global choice of gauge []. We can circumvent this problem in perturbation theory, since the Dyson series is defined in the neighborhood of a specific classical vacuum and is thus strictly local. We will encounter the nonperturbative breakdown of this loophole in Sec. 3. Given $F[A]$ unambiguously, we may impose the gauge condition by integrating over the space of gauge transformations $g \in G$ of A:

$$\int \mathcal{D}g\ \delta(F[gAg^{-1}]) \det \delta_g F = 1, \tag{1.40}$$

which is inserted into the path integral:

$$\int \mathcal{D}A\ e^{iS} = \int \mathcal{D}[A, g]\ e^{iS}\delta(F) \det \delta_g F, \tag{1.41}$$

Where gauge invariance allows us to ignore the gauge transformation within the delta function. The Jacobian determinant may be rephrased as a Gaussian path integral over some Grassmann-valued scalar fields $c = c^a t^a$ and $\bar{c} = \bar{c}^a t^a$ in the adjoint representation,

$$\det \delta_g F = \int \mathcal{D}[c, \bar{c}]\ \exp\left\{ i \int \operatorname{Tr} \bar{c}(\delta_g F)c \right\}, \tag{1.42}$$

producing a new contribution to the action:

$$\int \mathcal{D}A\, e^{iS} = \int \mathcal{D}[A, g, c, \bar{c}]\, e^{i(S+S_{FP})}\delta(F), \tag{1.43}$$

where

$$S_{FP} = \int \mathcal{L}_{FP}, \qquad \mathcal{L}_{FP} = \mathrm{Tr}\,\bar{c}(\delta_g F)c \tag{1.44}$$

is the Faddeev-Popov action. It defines two virtual, anticommuting scalar fields $c, \bar{c}$, called ghosts and antighosts, which exactly cancel the unphysical polarizations. The typical generalization of the Lorenz gauge condition defines the function

$$F[A] = \partial_\mu A^\mu - \omega, \tag{1.45}$$

for a smooth function ω. In this case, the Jacobian assumes the form

$$\delta_g F = \partial_\mu D^\mu, \tag{1.46}$$

where now the covariant derivative acts on the adjoint representation of G, where the ghosts assume values;

$$Dc = \partial c + [A, c]. \tag{1.47}$$

The arbitrary function ω may be removed from the path integral by integrating in ω the entire generating functional with a Gaussian weight, which scales the entire integral by some volume $1/N$:

$$\begin{aligned} \int \mathcal{D}A\, e^{iS} &= N \int \mathcal{D}\omega\, e^{\frac{i}{2\xi g^2}\int \mathrm{Tr}\,\omega^2} \int \mathcal{D}[A, g, c, \bar{c}]\, e^{i(S+S_{FP})}\delta(\partial_\mu A^\mu - \omega) \\ &= \int \mathcal{D}[A, g, c, \bar{c}]\, e^{i(S+S_{FP})} e^{\frac{i}{\xi}\int \mathrm{Tr}(\partial_\mu A^\mu)^2}. \end{aligned} \tag{1.48}$$

The resulting exponential contains another action defining the gauge-fixing Lagrangian,

$$\mathcal{L}_{gf} = \frac{1}{2\xi g^2}\,\mathrm{Tr}\big(\partial_\mu A^\mu\big)^2. \tag{1.49}$$

Now, the gauge is determined by the choice of some positive real scalar ξ. Gauges with this choice of the function F are known as renormalizable-ξ (R_ξ) gauges. We can see that as ξ tends to zero, finiteness of the action requires that $\partial_\mu A^\mu = 0$. This choice, called the Landau gauge condition, is equivalent to the classical Lorenz condition. The gauge-fixing Lagrangian contributes to the quadratic term in the gauge fields, so that now

$$\mathcal{L} \sim \int A^\nu \left[q^2 g_{\mu\nu} - \left(1 - \frac{1}{\xi}\right) q_\mu q_\nu\right] A^\mu, \tag{1.50}$$

which has an invertible kernel:

$$\langle A_\nu(-q)A_\mu(q)\rangle \sim \frac{1}{q^2}\left[g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right]. \tag{1.51}$$

We will generally work in the Feynman gauge, $\xi = 1$, since the gauge field propagator is particularly simple with this choice. Somehow, removing the redundant degrees of freedom also decouples the unphysical polarizations.

This may be explained through the notion of BRST (after Becchi, Rouet, Stora, and Tyutin []) symmetry, which remains unbroken even after gauge fixing. Notice that the gauge-fixing condition forces $\partial^2 \omega = 0$, which is precisely the form of the equations of motion for free ghosts.We might then imagine a gauge-like transformation generated by ghosts with some other constant Grassmann variable θ ensuring overall commutativity; this is the BRST transformation:

$$\phi \to \phi + \theta s \phi, \tag{1.52}$$

where s is called the Slavnov differential. To avoid using the equations of motion, we rearrange the Lagrangian in terms of an auxiliary Nakanishi-Lautrup field $B = B^a t^a$,

$$\mathcal{L}_{gf} = -\frac{1}{2} \xi g^2 \operatorname{Tr} B^2 + \operatorname{Tr} B \partial_\mu A^\mu, \tag{1.53}$$

which may be integrated out of the functional integral since it does not propagate:

$$\xi g^2 B - \partial_\mu A^\mu = 0 \Rightarrow \mathcal{L}_{gf} \sim \frac{1}{2\xi g^2} \operatorname{Tr}\big(\partial_\mu A^\mu\big)^2. \tag{1.54}$$

The Slavnov operator is defined so that the action on fermion and gauge fields goes as a gauge transformation (Eq. 1.5 and Eq. 1.7) generated by c with $\theta s \phi = \delta \phi$,

$$\delta \psi = -\theta c \psi, \quad \delta \bar{\psi} = -\theta \bar{\psi} c, \quad \delta A_\mu = \theta D_\mu c, \tag{1.55}$$

while ghosts transforms in such a way to maintain invariance of $D_\mu c^a$,

$$\delta c = -\theta c^2 = -\frac{1}{2} \theta f^{abc} t^a c^b c^c. \tag{1.56}$$

The antighosts cancel the variation of the gauge-fixing term,

$$\delta \bar{c} = -\theta B, \tag{1.57}$$

and the auxiliary field is unchanged,

$$\delta B = 0. \tag{1.58}$$

With the help of the Jacobi identity, the full gauge-fixed Lagrangian is found to be invariant under the action of s. Moreover, the BRST transformation is nilpotent,

$$s^2 \phi = 0, \tag{1.59}$$

for any field ϕ, so it determines a cohomology on the (pseudo-inner product) space of states. Since s generates a continuous symmetry, there is a conserved charge called the ghost number. Schematically, for the space Ω_n of states of ghost number n, the n^{th} BRST transformation maps

$$s : \Omega_n \to \Omega_{n+1}, \quad |a, n\rangle \mapsto |b, n+1\rangle \tag{1.60}$$

Since it is nilpotent, we have $\mathcal{B}_n \subset \mathcal{Z}_n$, where

$$\mathcal{Z}_n = \text{Ker}\left\{s : \Omega_n \to \Omega_{n+1}\right\}, \quad \mathcal{B}_n = \text{Im}\left\{s : \Omega_{n-1} \to \Omega_n\right\}, \tag{1.61}$$

and the space of states annihilated by s (closed states) is divided into equivalence classes determined by the n^{th} BRST cohomology group:

$$H^n_{BRST} = \frac{\mathcal{Z}_n}{\mathcal{B}_n}. \tag{1.62}$$

Kujo and Ojima showed that H^n_{BRST} contains the unique physical states []. These are states of the form

$$|a, n\rangle + s|b, n-1\rangle, \tag{1.63}$$

where $|a, n\rangle$ is closed. The additional exact term does not affect the BRST transformation, which reflects the symmetry of the action under gauge transformations. In the zero coupling limit, the variations above show that s converts antighosts to auxiliary fields (equivalent to (backwards) longitudinal gauge bosons) and converts gauge bosons to ghosts, so that the longitudinal bosons and ghosts are exact states, and thus have zero norm. There is an analogous anti-BRST homology that determines an equivalent condition on the antighost states. This demonstrates the relation between the redundant degrees of freedom in the path integral and the unphysical polarizations. All physically equivalent gauge configurations are related up to an exact form, which is associated with the unphysical (zero-norm) states of the theory. Intuitively, since the Faddeev-Popov ghosts are Grassmann-valued, their contribution to the path integral in d-dimensions goes as $\det\left(\partial^2\right)^d$. On the other hand, free gauge bosons are complex fields following the same wave equation, so they contribute $\det\left(\partial^2\right)^{-d/2}$. In four dimensions, the ghosts cancel exactly two degrees of freedom, the longitudinal and timelike polarizations.

1.6 Renormalization

In a free field theory, once the two-point correlation function is known, the theory is solved. One may determine the propagation amplitude for a single particle over any spacetime interval, which corresponds to an inner product in the Fock space of free eigenstates. As interactions are introduced, however, there are excitations for any number of particles, and the propagator loses its intuitive interpretation. The eigenspace of the interacting Hamiltonian is, per Haag's theorem, unitarily inequivalent to the free case, so that no isomorphism may be found between the free and interacting Hilbert spaces. This section introduces renormalization, which circumvents the assumptions of Haag's theorem, and allows us to loosely construct a space of interacting multiparticle states. We may analyze the spectrum of the propagator in the interacting theory by inserting the completeness relation on the new Hilbert space. Since momentum is conserved, Hamiltonian eigenstates are simultaneously momentum eigenstates, and we may express the sum over states as a sum over zero-momentum eigenstates $|\lambda, 0\rangle$ integrated over all boosts $|\lambda, \mathbf{p}\rangle$ of the resting states:

$$\mathbb{1} = \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\lambda, \mathbf{p})} |\lambda, \mathbf{p}\rangle\langle\lambda, \mathbf{p}|, \tag{1.64}$$

where $\omega(\lambda, \mathbf{p}) = \sqrt{\mathbf{p}^2 + m_\lambda^2}$ is the energy of the state $|\lambda, \mathbf{p}\rangle$ with physical mass m_λ. For simplicity, we specialize to ϕ_0^4 theory with the Lagrangian[1]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_0)(\partial^\mu \phi_0) - \frac{1}{2} m_0^2 \phi_0^2 - g_0 \phi_0^4, \tag{1.65}$$

where the free propagator[2] is given by

$$\tilde{S}_0^{(0)}(p, m_0) = \frac{i}{p^2 - m_0^2 + i\epsilon}. \tag{1.66}$$

We now insert Eq. 1.64 into the (time-ordered) two-point function:

$$\langle 0|\phi_0(y)\phi_0(x)|0\rangle = \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip(x-y)}}{2\omega(\lambda, \mathbf{p})} |\langle 0|\phi_0(0)|\lambda, 0\rangle|^2. \tag{1.67}$$

Under time ordering, we uncover the causal propagator for mass m_λ,

$$\langle 0|\mathcal{T}\phi_0(y)\phi_0(x)|0\rangle = \sum_\lambda S(x-y, m_\lambda)|\langle 0|\phi_0(0)|\lambda, 0\rangle|^2 = \int_0^\infty \frac{dM}{2\pi}\rho(M)S(x-y, M). \tag{1.68}$$

This is the Källén-Lehmann spectral representation [], with the spectral density

$$\rho(M) = \sum_\lambda 2\pi\delta(M - m_\lambda)|\langle 0|\phi_0(0)|\lambda, 0\rangle|^2 = 2\pi\delta(M - m_\phi) \cdot Z_\phi + \cdots, \tag{1.69}$$

where $Z_\phi = |\langle 0|\phi_0(0)|1, 0\rangle|^2$ ($\lambda = 1$ is a shorthand for a one-particle state), m_ϕ is the mass of the single-particle state, and the truncated terms represent multiparticle states[3]. Passing to momentum space, we may finally write

$$\langle 0|\mathcal{T}\tilde{\phi}_0(-p)\tilde{\phi}_0(p)|0\rangle = \frac{iZ_\phi}{p^2 - m_\phi^2 + i\epsilon} + \cdots. \tag{1.70}$$

Z_ϕ is the vacuum expectation value of a single particle state including all self-interactions. We may absorb it into the normalization of the fields by defining $\phi_0 = Z_\phi^{1/2}\phi$, so that

$$\langle 0|\mathcal{T}\tilde{\phi}(-p)\tilde{\phi}(p)|0\rangle = \frac{i}{p^2 - m_\phi^2 + i\epsilon} + \cdots, \tag{1.71}$$

Here, the renormalized field ϕ now accounts for all of the quantum fluctuations induced by interactions. The mass m_ϕ in Eq. 1.71 is the physical mass of a single-particle state in the

[1]The subscript zeroes are written here for "bare" quantities in anticipation of later results.

[2]The term $i\epsilon$ is an infinitesimal imaginary regulator maintaining causality which is sent to zero in practice. ADD NEW CONTOUR PLOT AND BRIEFLY DISCUSS ROTATION

[3]We have also quietly discarded the constant – typically vanishing – contribution from the ground state. In ϕ^4 theory, this term vanishes by Lorentz invariance

interacting theory, since it is the eigenvalue of the squared momentum operator. This is to be contrasted with the parameter m_0 in the Lagrangian, which corresponds to the mass of the free theory. Evidently, in the presence of interactions, the pole of the propagator is shifted from m_0^2 to m_ϕ^2. This is the essence of renormalization; the inclusion of interactions in a field theory perturbs physical, measurable quantities away from the bare parameters of the Lagrangian.

We may similarly define the renormalized mass and coupling through $Z_m m^2 = Z_\phi m_0^2$ and $Z_g g = Z_\phi^2 g_0$, so that the Lagrangian may be cleanly rewritten as

$$\mathcal{L} = \frac{1}{2} Z_\phi (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} Z_m m^2 \phi^2 - Z_g g \phi^4. \tag{1.72}$$

Since the free theory undergoes no renormalization due to quantum fluctuations, all parameters in the bare Lagrangian (1.65) are physical Then for each renormalization constant Z_i we have the decomposition

$$Z_i = 1 + \delta_i, \tag{1.73}$$

where $\delta_i = \mathcal{O}(g_0)$ vanishes as the coupling goes to zero. This allows us to write the renormalized Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - g\phi^4 + \frac{1}{2}\delta_\phi (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}\delta_m m^2 \phi^2 - \delta_g g \phi^4. \tag{1.74}$$

The last three terms above are called counterterms. They each produce analogous vertices to those of the bare Lagrangian, and they will prove useful in the next section.

At this point, there is a proliferation of unknowns in the Lagrangian. We must relate the bare, renormalized, and physical parameters. In general, in order to calculate the renormalized parameters, $\{a_i\}$, we need as many equations, called renormalization conditions, relating them to calculable functions $\{f_i\}$ of the bare parameters $\{a_i^0\}$:

$$a_i = f_i(a_1^0, a_2^0, ...). \tag{1.75}$$

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1.7 Regularization

In the perturbative expansion, as the interaction Lagrangian is successively inserted into a correlation function, the increasing number of virtual contractions leads to closed loops of contracted vertices. In the momentum space representation, the spacetime integral associated with each insertion generates a delta distribution over the sum of all ingoing momenta, corresponding to a conservation of momentum at that vertex. In a loop with some n vertices, the momenta of any $n-1$ internal propagators fix the value of the n^{th} momentum, so that there is an overall integral over the total loop momenta. For example, in ϕ^4 theory the first loop correction appears at $\mathcal{O}(g_0)$:

$$g_0 \tilde{S}_0^{(1)}(p) = (\text{scalar self-energy}) = g_0 \frac{i}{p^2 - m_0^2 + i\epsilon} \left[-\frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m_0^2 + i\epsilon} \right] \frac{i}{p^2 - m_0^2 + i\epsilon}, \tag{1.76}$$

where $\tilde{S}_0^{(1)}$ is the next-to-leading order (NLO) contribution to the perturbation expansion of the bare propagator:

$$\tilde{S}_0 = \sum_{n=0}^{\infty} g_0^n \tilde{S}_0^{(n)}. \tag{1.77}$$

The integral above diverges quadratically as the loop momenta becomes infinite, as may be seen by transforming to Euclidean (Sec. ??) spherical coordinates so that the magnitude of the loop momentum is explicit:

$$\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_0^2 + i\epsilon} = \lim_{\Lambda\to\infty} \frac{1}{8\pi^2} \int_0^\Lambda dk \frac{k^3}{k^2 + m_0^2} = \lim_{\Lambda\to\infty} \frac{m_0^2}{(4\pi)^2} \left[\frac{\Lambda^2}{m_0^2} - \log\left(1 + \frac{\Lambda^2}{m_0^2}\right)\right]. \tag{1.78}$$

discuss $+i\epsilon$ and W.R. This form of the integral, though formally infinite, demonstrates the effect of a hard cutoff on the loop momentum. If we drop the limit above, we may interpret our theory to be only well-defined in the infrared region $0 \leq k^2 \leq \Lambda^2$ up to some threshold scale Λ. This is an elementary example of regularization, where a divergent integral is recast as the limit of some divergent sequence of finite integrals.

The cutoff regularization above breaks gauge symmetry, so it is rarely useful in practice. Instead, divergent Feynman integrals are typically treated in dimensional regularization, wherein the spacetime dimension is analytically continued away from four dimensions to a generically complex value d. In order to calculate integrals in d dimensions, we must transform to a coordinate system where some number of dimensions may be integrated directly by symmetry. The easiest and most common practice is to find some parametrization such that the integrand depends solely on the magnitude of the loop momentum. In this case, the integrand is made spherically-symmetric, and the $(d-1)$-dimensional solid angle integral may be read off, leaving only the radial integral:

$$\int \frac{d^dk}{(2\pi)^d} f(k^2) = \frac{1}{(2\pi)^d} \int d\Omega_{d-1} \int_0^\infty dk\ k^{d-1} f(k^2) = 2\frac{(4\pi)^{2-d/2}}{\Gamma(d/2)} \int_0^\infty dk\ k^{d-1} f(k^2). \tag{1.79}$$

Clearly, now the degree of divergence of the integrand is dependent on the dimension, which is customarily written as $d = 4 \pm 2\epsilon$, where ϵ is the radius of some neighborhood of the physical value of $d = 4$. After the momentum integral is calculated, the result is expanded in a Laurent series about $\epsilon = 0$. Terms which go as inverse powers of ϵ represent the same divergences as in cutoff regularization. In the example above, the dimensionally-regularized integral is

$$-\frac{g_0}{2} \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2 - m_0^2} = \frac{g_0}{2} \frac{m_0^2}{(4\pi)^2} \left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{m_0^2}\right) - \gamma_E + 1 + \mathcal{O}(\epsilon)\right]. \tag{1.80}$$

Since the integration measure is now d dimensional, we account for this deviation by changing the canonical dimension of the coupling: $[g_0] = 4 - d$, and modifying the renormalization condition

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; this will eventually compensate for the awkward dimensionful logarithm.

Regardless of the chosen regulator, we may now write the NLO propagator:

$$\tilde{S}_0(p) = \tilde{S}_0^{(0)}(p) + g_0\tilde{S}_0^{(1)}(p) + \mathcal{O}(g_0^2) = \frac{i}{p^2 - m_0^2} + g_0\frac{i}{p^2 - m_0^2}\Sigma^{(1)}\frac{i}{p^2 - m_0^2} + O(g_0^2), \quad (1.81)$$

where $\Sigma^{(1)}$ represents an appropriately regularized version of the bracketed expression in Eq. 1.76. Continuing to higher orders, this sum easily seen to be a geometric series, where $\Sigma^{(n)}$ represents the one-particle-irreducible (1PI) correlation function of order n; *viz.*, the sum of all Feynman diagrams at $\mathcal{O}(g_0^n)$ whose graphs contain no bridges. Chaining these together by tree-level bridges gives us the series

$$\tilde{S}(p) = \frac{i}{p^2 - m_0^2}\sum_{n=0}^{\infty}\left(\Sigma\frac{i}{p^2 - m_0^2}\right)^n = \frac{i}{p^2 - m_0^2 - \Sigma} = \frac{i}{p^2 - m_0^2 - g_0\Sigma^{(1)} + \mathcal{O}(g_0^2)}. \quad (1.82)$$

In both regularization schemes above, $\Sigma^{(1)} \propto m_0^2$, so that it can be absorbed into some shift in the mass $\delta_m^{(1)}$:

$$\tilde{S}(p) = \frac{i}{p^2 - m_0^2(1 - g_0\delta_m^{(1)}) + \mathcal{O}(g_0^2)}. \quad (1.83)$$

In dimensional regularization, we find

$$\delta_m^{(1)} = -\frac{1}{2}\frac{m_0^2}{(4\pi)^2}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{m_0^2}\right) - \gamma_E + 1 + \mathcal{O}(\epsilon)\right]. \quad (1.84)$$

This suggests that the mass m_0 in the Lagrangian and the mass satisfying the Klein-Gordon equation of motion differ in the interacting theory. The infinite shift from m_0 in the physical mass represents the energy of the particle interacting with its own field, the self-energy of ϕ_0, which receives contributions from infinitely many loop corrections. Since the measured value is obviously finite, the “bare” mass m_0 must compensate for the infinite shift. We reinterpret the bare mass as the (potentially infinite) rest mass of a single-particle excitation of ϕ_0, dressed with all virtual self-energy interactions. The pole mass is then given by

$$m^2 = m_0^2 Z_m^{-1} Z_\phi = m_0^2 Z_\phi(1 - \delta_m), \quad (1.85)$$

where the infinite corrections in δ_m exactly cancel those included in the redefinition of the bare mass.

At the next order in the coupling, there are momentum-dependent terms which diverge as the regulator is removed, which we represent by $p^2\delta_\phi$. Resumming these contributions, the two-point function may be written as

$$\tilde{S}(p) = \frac{i}{p^2(1 - \delta_\phi) - m_0^2(1 - \delta_m)} = \frac{iZ_\phi}{p^2 - m^2}, \quad (1.86)$$

where the second equivalence enforces the Källén-Lehmann representation. We may then read off $Z_i^{-1} = 1 - \delta_i$ for both the mass and field renormalizations.

1.8 Effective Field Theory

1.9 The Renormalization Group

1.10 Operator Mixing

Chapter 2

Quantum Chromodynamics

2.1 Yang-Mills Theory

Quantum chromodynamics is the gauge theory for the strong interaction. It is easily constructed from only a few empirical principles. We begin the fermionic Lagrangian, describing a set of uncoupled spin-1/2 quarks with $n_f = 6$ flavors:

$$\mathcal{L}_f = \sum_{i=1}^{n_f} \bar{\psi}_i (i\partial\!\!\!/ - m_i)\psi_i \tag{2.1}$$

From now on the sum over flavors will be assumed, and we will drop the corresponding indices. The interacting theory is determined by imposing a local gauge symmetry with the condition of renormalizability. In order to choose a symmetry group, we observe that the branching fractions for mounic and hadronic decay channels in electron-positron strongly suggest that quarks compose a triplet representation of their gauge group []. Since $SO(3)$ and $SU(3)$ are the only compact, simple Lie groups up to isomorphism with three-dimensional irreps, it must be one of these two. However, quarks cannot be their own antiparticles, so we require a complex representation, which exclusively establishes $SU(3)$ as the gauge group. The fundamental triplet representation is defined over a vector space of three basis "color" charges held by the quarks. For full generality in choosing the number of colors N, we instead study $SU(N)$ for the remainder of this text. Some properties of $SU(N)$ and its Lie algebra are discussed in App. **??**.

Now, as in Sec. 1.1, in order that the Lagrangian maintain invariance under local gauge transformations, the derivative term must be made to transform covariantly. The true reason for our earlier construction is the locality of the transformation. Since this means ω is coordinate-dependent, the infinitesimally-separated fields under the derivative transform separately, and we need to consider their parallel transport on a path across the displacement:

$$\delta\psi = A_\mu \psi. \tag{2.2}$$

where A_μ is the gluon field, which assumes values in the Lie algebra $\mathfrak{su}(3)$. Adding this to the naïve derivative yields the gauge covariant derivative,

$$D_\mu \psi = (\partial_\mu + A_\mu)\psi, \tag{2.3}$$

which transforms as desired; *viz.*, for some $U \in SU(3)$,

$$D_\mu \psi \xrightarrow{U} D'_\mu \psi' = U D_\mu \psi. \tag{2.4}$$

Since $SU(N)$ is a matrix Lie group, the covariant derivative acts on fields in the fundamental representation – fermions in the present case – by multiplication as in Eq. 2.3. The gluons instead assume the adjoint representation, for which the connection is simply $ad_A(\cdot) = [A, \cdot]$, so the covariant derivative acts accordingly:

$$D_\mu A_\nu = \partial_\mu A_\nu + [A_\mu, A_\nu] = \left(\partial_\mu A^a_\nu + f^{abc} A^b_\mu A^c_\nu\right) t^a. \tag{2.5}$$

The replacement $\partial \to D$ adds an interaction piece to the fermionic Lagrangian,

$$\mathcal{L}_{int} = i\bar{\psi}_i \not{A} \psi, \tag{2.6}$$

and may be identified with the minimal coupling prescription. We have now traded gauge variance for a new vector field, A, which further requires its own free Lagrangian in order to be dynamical. The Proca Lagrangian,

$$\mathcal{L}_P = \frac{1}{g^2} \operatorname{Tr}\left[\frac{1}{2} G_{\mu\nu} G^{\mu\nu} - M^2 A_\mu A^\mu\right], \tag{2.7}$$

where $G_{\mu\nu} = [D_\mu, D_\nu]$, describes free vector particles of mass M. The tensor $G_{\mu\nu} = G^a_{\mu\nu} t^a$ is known as the field-strength (curvature) tensor for the field A. The mass term in the above Lagrangian is not gauge invariant; setting $M = 0$, we arrive at the Yang-Mills Lagrangian:

$$\mathcal{L}_{YM} = \frac{1}{2g^2} \operatorname{Tr} G_{\mu\nu} G^{\mu\nu}. \tag{2.8}$$

We now have the nonperturbative QCD Lagrangian, containing kinetic terms for both massive fermions and massless gauge bosons — respectively the quarks and gluons — and a minimal interaction term between them:

$$\mathcal{L}_{QCD} = \mathcal{L}_f + \mathcal{L}_{YM} + \mathcal{L}_{int} = \bar{\psi}(i\not{D} - m)\psi + \frac{1}{2g^2} \operatorname{Tr} G_{\mu\nu} G^{\mu\nu}. \tag{2.9}$$

The gluon Lagrangian also hides two self-interactions generated by nonlinearities in the field-strength tensor

$$G^a_{\mu\nu} = \partial_{[\mu} A^a_{\nu]} + f^{abc} A^b_\mu A^c_\nu. \tag{2.10}$$

Because the gauge group in nonabelian, the commutator is generically nonzero, and the product $G_{\mu\nu} G^{\mu\nu}$ contains quadratic, cubic, and quartic interactions.

From here, the generating functional produces all correlation functions:

$$\left\langle \prod_{i=1}^{n_G} G_{\mu_i}(z_i) \prod_{j=1}^{n_\psi} \psi(y_j) \prod_{k=1}^{n_\psi} \bar{\psi}(x_k) \right\rangle = \prod_{i=1}^{n_G} \frac{-i\delta}{\delta J^{\mu_i}(z_i)} \prod_{j=1}^{n_\psi} \frac{i\delta}{\delta\bar{\eta}(y_j)} \prod_{k=1}^{n_\psi} \frac{-i\delta}{\delta\eta(x_k)} \frac{1}{Z_0} \int \mathcal{D}[A, \psi, \bar{\psi}] e^{iS} \bigg|_{\bar{\eta},\eta,J=0}, \tag{2.11}$$

where we have inserted the appropriate numerical factors respecting fermionic statistics and modified the action to include the appropriate sources, as in Sec. 1.2:

$$S = \int d^4x \left[\mathcal{L}_{QCD} + J_\mu A^\mu + \bar{\psi}\eta + \bar{\eta}\psi\right]. \tag{2.12}$$

Here, in accordance with the fields they source, the J field is a Lorentz vector taking values in the adjoint representation of $SU(3)$, while the $\eta, \bar{\eta}$ fields are Grassmann-valued spinors.

In order to study QCD perturbatively, we must fix the gauge. Following the Faddeev-Popov procedure as before, we introduce two new terms to the action for the Faddeev-Popov and gauge-fixing Lagrangians (Eqs. 1.44 and 1.49) defined in an R_ξ gauge. The total Lagrangian is thus

$$\mathcal{L}_{QCD} = \mathcal{L}_D + \mathcal{L}_{YM} + \mathcal{L}_{gf} + \mathcal{L}_{FP} + \mathcal{L}_J, \tag{2.13}$$

where

$$\mathcal{L}_D = \sum_{i=1}^{n_f} \bar{\psi}_i (i\not{D} - m_i)\psi_i, \tag{2.14a}$$

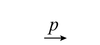
$$\mathcal{L}_{YM} = \frac{1}{2g^2}\operatorname{Tr} G_{\mu\nu}G^{\mu\nu}, \tag{2.14b}$$

$$\mathcal{L}_{gf} = \frac{1}{2\xi g^2}\operatorname{Tr}\left(\partial_\mu A^\mu\right)^2, \tag{2.14c}$$

$$\mathcal{L}_{FP} = \operatorname{Tr}\bar{c}(\partial_\mu D^\mu)c, \tag{2.14d}$$

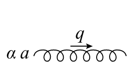
$$\mathcal{L}_J = J_\mu A^\mu + \bar{\psi}\eta + \bar{\eta}\psi + \bar{c}\kappa + \bar{\kappa}c, \tag{2.14e}$$

and we have introduced the Grassmann-odd, scalar source fields $\kappa, \bar{\kappa}$ for the ghosts. We may now construct the two-point Green's functions, or propagators, for the fermions and gluons. Since we are calculating two-point functions, the only contributions at leading order in the coupling come from the kinetic part of the action, which is strictly quadratic in the fields. Following the Gaussian integration procedure in App. B, we have three propagators. For now we simply write the matrix inverse of each particle species' kinetic operator. For quarks, the momentum-space Dirac operator reads $-\not{p} - m$, and the inverse is

$\langle\tilde{\psi}\tilde{\bar{\psi}}\rangle$: 

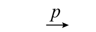
$$= \tilde{S}^{(0)}(p) = \frac{-\not{p} + m}{p^2 - m^2 + i\epsilon}. \tag{2.15a}$$

We have already considered the gluon propagator in Eqs. 1.50 and 1.51. The propagator is

$\langle\tilde{A}\tilde{A}\rangle$: 

$$= [\tilde{D}^{ab}_{\alpha\beta}]^{(0)}(q) = \frac{1}{q^2 + i\epsilon}\left[g_{\alpha\beta} - (1-\xi)\frac{q_\alpha q_\beta}{q^2}\right]. \tag{2.15b}$$

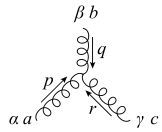
Since the ghosts obey the Laplace (or Poisson for nonzero $i\epsilon$) equation, their propagator is the well-known fundamental solution,

$\langle\tilde{c}\tilde{\bar{c}}\rangle$: 

$$= -\frac{1}{p^2 + i\epsilon}. \tag{2.15c}$$

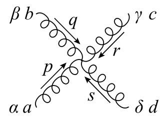
There are also four vertices involving higher powers of the fields. As we mentioned before, the Yang-Mills action contains vertices with three and four gluons:

$\langle \tilde{A}^3 \rangle$:



$$= \frac{if^{abc}}{g^2}\left[(p-q)_\gamma g_{\alpha\beta} + (q-r)_\alpha g_{\beta\gamma} + (r-p)_\beta g_{\gamma\alpha}\right], \quad (2.16a)$$

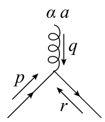
$\langle \tilde{A}^4 \rangle$:



$$\begin{aligned} = \quad & f^{abe}f^{cde}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\gamma\beta}) \\ & + f^{ace}f^{bde}(\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\delta}\delta_{\gamma\beta}) \\ & + f^{ade}f^{bce}(\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\gamma}\delta_{\beta\delta}). \end{aligned} \quad (2.16b)$$

The quark-quark-gluon vertex arises as a result of promoting the derivative to a covariant derivative, as in Sec. 1.1:

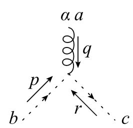
$\langle \tilde{\psi}\tilde{A}\tilde{\overline{\psi}} \rangle$:



$$= -\gamma_\alpha t^a. \quad (2.16c)$$

Likewise, at nonzero coupling the covariant derivative in the Faddeev-Popov action generates a ghost-ghost-gluon vertex:

$\langle \tilde{c}\tilde{A}\tilde{\overline{c}} \rangle$:



$$= -if^{abc}r_\alpha. \quad (2.16d)$$

2.2 Renormalization

Now that we have the Feynman rules for QCD, we can in principle calculate correlation functions to any order – at least in terms of momentum integrals. Nevertheless, one encounters difficulties already at next-to-leading order.[1] After the tree-level, new and more vertices appear within the diagrams, forming loops. These fluctuations correspond to the mixing of all couplings in the Lagrangian. Hence, these diagrams represent the renormalizations of the bare fields and parameters. QCD contains six bare parameters: the normalizations of the fermion, gluon, and ghost wavefunctions; the strong coupling g; the fermion mass m, and the gauge-fixing parameter ξ. The gauge-fixing parameter requires renormalization in order to consistently maintain gauge invariance. Briefly, we consider the geometric series for the gluon propagator (see Eqs. 1.82 and 2.15b),

$$\tilde{D}^{ab}_{\alpha\beta}(q) = [\tilde{D}^{ab}_{\alpha\beta}]^{(0)}(q) + [\tilde{D}^{ac_1}_{\alpha\mu_1}]^{(0)}(q)\tilde{\Pi}^{c_1 d_1}_{\mu_1\nu_1}(q)[\tilde{D}^{d_1 b}_{\nu_1\beta}]^{(0)}(q) + \cdots, \quad (2.17)$$

where $\tilde{\Pi}^{ab}_{\alpha\beta}(q)$ represents the sum over all 1PI diagrams in the propagator of a gauge boson. Lorentz invariance and the conservation of gluon charge restricts its form to $\delta^{ab}\left(\pi_1\delta_{\alpha\beta} - \pi_2\frac{q_\alpha q_\beta}{q^2}\right)$

[1] Except for vacuum correlation functions of nontrivial operators, whose leading order diagrams are already one-loop

for some functions $\pi_{1,2}(q)$. According to the transformation law for A, Eq. **??**, a gauge transformation shifts any physical state by a total divergence that must identically vanish in order to uphold gauge invariance. In momentum space, this is the Slavnov-Taylor (ST) identity

$$q_\alpha \tilde{\Pi}^{ab}_{\alpha\beta}(q) = 0 \tag{2.18}$$

Gauge invariance then additionally requires that $\pi_1 = \pi_2$, so that only the transverse polarizations of the gluon propagator receive loop corrections. Since the gauge-fixing term is entirely longitudinal, we must ensure that it does not acquire an anomalous dimension. If we write

$$g_0 = \mu^\epsilon Z_g g, \qquad \xi_0 = Z_\xi \xi, \qquad A_0 = Z_A^{1/2} A, \tag{2.19}$$

the corresponding renormalized Lagrangian goes as $\frac{Z_A}{Z_\xi Z_g^2}$, which is restricted to one by the ST identity. This evidently allows us to discard one of these constants. To make later manipulations cleaner, we choose $Z_A = Z_\xi Z_g^2$.

2.3 Euclidean QCD

Chapter 3

Lattice Quantum Chromodynamics

INTRO

3.1 The Simplest Gauge Action

We define the $n \times m$ Wilson loop $W_{\mu\nu}^{n\times m}(x)$ as the path-ordered product of gauge links around a closed convex loop. This $SU_3(\mathbb{C})$ matrix encodes the holonomy of the gauge covariant derivative around discretized curves on a $4-D$ lattice. The simplest case, a so-called plaquette $W_{\mu\nu}^{1\times 1}(x)$, is given by

$$W_{\mu\nu}^{1\times 1}(x) \equiv U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu^\dagger(x+a\hat{\nu})U_\nu^\dagger(x). \tag{3.1}$$

We may expand this product explicitly, keeping only terms up to order a^2, where a is the lattice spacing:

$$\begin{aligned} W_{\mu\nu}^{1\times 1}(x) &= e^{igaA_\mu(x)}e^{igaA_\nu(x+a\hat{\mu})}e^{-igaA_\mu(x+a\hat{\nu})}e^{-igaA_\nu(x)} \\ &= 1 + iga^2G_{\mu\nu}(x) - g^2a^4G_{\mu\nu}G_{\mu\nu}(x) + \mathcal{O}(a^6). \end{aligned} \tag{3.2}$$

The first line expresses the gauge links as exponentials. The second line expands pairwise the products of the exponentials in terms of the Baker-Campbell-Hausdorff (BCH) formula to second order in the lattice spacing. The third line expands the gauge potentials A about x to first order in a. This avoids more onerous commutators, as it allows us to immediately identify and ignore any higher order terms in the subsequent BCH expansion. The fourth line uses the BCH formula again to expand the final product, this time absorbing any cubic terms into the $\mathcal{O}(a^3)$ term, which cancel as we expand the exponential.

We may now express the lattice gauge action in terms of the plaquette $W_{\mu\nu}^{1\times 1}(x)$:

$$S_G[U] \equiv \frac{2}{g^2}\sum_x\sum_{\mu<\nu}\mathfrak{Re}\mathrm{Tr}\big\{1 - W_{\mu\nu}^{1\times 1}(x)\big\} = \frac{1}{2}a^4\sum_x\sum_{\mu,\nu}\Big[\mathrm{Tr}\big\{G_{\mu\nu}G_{\mu\nu}\big\} + \mathcal{O}(a^2)\Big]. \tag{3.3}$$

In the limit of zero lattice spacing, the sum over lattice points becomes an integral in $4-D$ Euclidean space with volume a^4; videlicet, $a^4\sum_x \to \int d^4x$ as $a \to 0$. Thus the lattice and continuum gauge actions are equal in the continuum limit:

$$S_G[U] \approx \frac{1}{2}a^4\sum_x\sum_{\mu,\nu}\mathrm{Tr}\big\{G_{\mu\nu}G_{\mu\nu}\big\} \xrightarrow{a\to 0} \frac{1}{4}\int_{\mathbb{R}^4} d^4x\ G_{\mu\nu}^aG_{\mu\nu}^a \equiv S_G[A]. \tag{3.4}$$

3.2 The Naïve Dirac Action

The simplest discretization of the Dirac operator involves replacing the derivative with a symmetrized finite difference quotient and inserting gauge links to restore gauge invariance:

$$\begin{aligned} S_F^N[\psi,\bar{\psi},U] &\equiv a^4 \sum_x \Big\{ \frac{1}{2a} \sum_\mu \bar{\psi}(x)\gamma_\mu \big[U_\mu(x)\psi(x+a\hat{\mu}) - U_{-\mu}(x)\psi(x-a\hat{\mu})\big] + m\bar{\psi}(x)\psi(x) \Big\} \\ &\xrightarrow{a\to 0} \int_{\mathbb{R}^4} d^4x \ \bar{\psi}\Big(\not{D} + m\Big)\psi \\ &= S_F[\psi,\bar{\psi},U]. \end{aligned} \tag{3.5}$$

This may be recast more compactly by noting that x is quantized by the lattice spacing: $x = na$, which allows us to condense the gauge link dependence into a linear combination of Kronecker delta functions. Define

$$\mathcal{M}_{xy}^N \equiv \frac{1}{2a} \sum_\mu \big[\gamma_\mu (U_\mu)_x \delta_{x,y-a\hat{\mu}} - \gamma_\mu (U_\mu)_{x-a\hat{\mu}} \delta_{x,y+a\hat{\mu}}\big] + m\delta_{x,y}. \tag{3.6}$$

Then the above reduces to

$$S_F^N[\psi,\bar{\psi},U] = a^4 \sum_{x,y} \bar{\psi}_x \mathcal{M}_{xy}^N \psi_y. \tag{3.7}$$

This form is conducive to faster computation, especially when the fields have been previously generated at all lattice sites. The delta functions simply project out the values at the relevant points on the lattice.

A problem arises, however, when constructing the free fermion propagator from this action. Let us transform to momentum space to illustrate this, turning off the gauge potentials to restore the free particle action ($U \to 1$):

$$\begin{aligned} S_F^0[\psi,\bar{\psi}] &= a^4 \sum_{x=na} \bar{\psi}(na) \Big\{ \frac{1}{2a} \sum_\mu \gamma_\mu \big[\psi(na+a\hat{\mu}) - \psi(na-a\hat{\mu})\big] + m\psi(na) \Big\} \\ &= a^4 \int_{-a/\pi}^{a/\pi} \frac{dk}{(2\pi)^4} \tilde{\bar{\psi}}(k) \Big\{ \frac{i}{a} \sum_\mu \gamma_\mu sin\big(k_\mu a\big) + m \Big\} \tilde{\psi}(k). \end{aligned} \tag{3.8}$$

Thus, we have constructed the momentum-space propagator for naïve lattice fermions:

$$S_F(k,m) = \frac{i}{a} \sum_\mu \gamma_\mu sin\big(k_\mu a\big) + m. \tag{3.9}$$

In the chiral limit, the inverse propagator has roots at the origin and the boundaries of the first Brillouin zone ($x_p \in \{0, a/\pi\}$), so there are $2^{D=4} = 16$ poles which characterize the famous "doublers" problem. Various correction schemes and systematic limitations of discretization will be discussed later.

3.3 The Haar Measure

By construction, the QCD action is invariant under gauge transformations:

$$S[U] = S[U']. \tag{3.10}$$

Reasonably, we require also that any observables are gauge invariant, so the functional integral

$$Z = \int D[U]e^{-S[U]} \tag{3.11}$$

must be gauge invariant. This is tantamount to invariance under a change of variables. This restriction on the path integral translates to demanding invariance under the action of the gauge group G of the integral measure $D[U]$ for any measurable subset $U \subseteq G$. The Haar measure satisfies this requirement naturally. We will construct this explicitly. For any locally compact T_2 group G, let us define the left translate of a Borel set $U \in \sigma(G)$:

$$gU = \{gu \mid u \in U\}, \text{ for some } g \in G. \tag{3.12}$$

Intuitively, this object should be the same "size" as the untranslated set. The goal is to find some measure $\mu(U)$, such that left translation by an element of the enclosing group does not affect this size; this is the Haar measure. We now state Haar's existence and uniqueness theorem for such a measure:

> *There exists a nontrivial, additive, regular measure on the Borel subsets of a locally compact Hausdorff group which is unique up to normalization, finite over compact sets, and invariant under left translation.*

Such a measure is called the left Haar measure:

$$\mu(U) = \mu(gU). \tag{3.13}$$

The details of the proof are beyond the scope of this thesis, but we may nonetheless apply the result to the gauge group in concern, $SU_3(\mathbb{C})$. Moreover, not only does there exist a left Haar measure in our case, but so also a right Haar measure, owing to the unimodularity of all compact Lie groups, $SU_N(\mathbb{C})$ included. We have, then, that

$$\mu(gU) = \mu(U) = \mu(Ug) \tag{3.14}$$

for any $g \in SU_N(\mathbb{C})$ with $U \in \sigma(G)$. Equipped with a measure, we may now consider integrals over locally compact groups. The invariance of the measure immediately implies the invariance of Lesbegue integrals when the integration variable is translated:

$$\int_U d\mu(u)f(u) = \int_U d\mu(gu)f(gu) = \int_U d\mu(u)f(gu) = \int_U d\mu(u)f(ug) \tag{3.15}$$

for some $u \in U$. This relationship is instrumental in finding exact solutions to many group integrals without requiring an explicit form of the Haar integral measure in terms of coordinates of the underlying manifold (odd sphere bundles in the case of $SU_N(\mathbb{C})$). We present a few useful results now:

$$\int_U d\mu(u) u_{ab} = 0 \tag{3.16}$$

$$\int_U d\mu(u) u_{ab} u_{cd} = 0 \tag{3.17}$$

$$\int_U d\mu(u) u_{ab} u^{\dagger}_{cd} = \delta_{ad}\delta_{bc} \tag{3.18}$$

$$\int_U d\mu(u) f(u) = \int_U d\mu(gu) f(gu) = \int_U d\mu(u) f(gu) = \int_U d\mu(u) f(ug) \tag{3.19}$$

Chapter 4

The Neutron Electric Dipole Moment

4.1 Baryon Asymmetry

4.2 Sakharov Conditions

4.3 Beyond the Standard Model Sources of CP-violation

4.4 The CPT Theorem

4.5 EDMs on the Lattice

Part II

The Gradient Flow

Chapter 5

The Flowed Formalism

The gradient flow belongs to a class of parabolic partial differential equations called geometric flows. In general, these equations describe the diffusion of some geometric quantity on a manifold. In particular, gradient flows in QFT are nonlinear heat equations on the space of configurations of a gauge field ϕ, which characterize its diffusion along some new dimension, the flow time t. Critically, this gives us the boundary condition that for $t = 0$ the flowed field Φ should coincide with the physical field ϕ. In order that the evolution is stable, the field should flow toward a local minimum of the action. This is accomplished by writing

$$\partial_t \Phi(x;t) = -\frac{\delta S[\Phi]}{\delta \Phi}, \qquad \Phi(x;0) = \phi(x), \tag{5.1}$$

where $S[\phi]$ is the action associated to the field ϕ at $t = 0$. Since the right side is proportional to the negative gradient of the action, we are assured that increasing the flow time drives the action toward a minimum as quickly as possible. We will find that this corresponds to a smearing of the gauge field in spacetime that suppresses ultraviolet modes.

5.1 The Yang-Mills Gradient Flow

In the case of the Yang-Mills action, Eq. **??**, it is straightforward to verify that

$$\frac{\delta S_{YM}[A]}{\delta A^a_\mu} = -(D_\nu G_{\nu\mu})^a. \tag{5.2}$$

Plugging this into the schematic equation (5.1) above, we arrive at a prototypical Yang-Mills gradient flow equation:

$$\partial_t B_\mu = D_\nu G_{\nu\mu}[B], \qquad B_\mu\big|_{t=0} = A_\mu, \tag{5.3}$$

where B is the flowed counterpart of the gluon field A defined in the bulk of the $(d+1)$-dimensional half-space with coordinates $(x; t \geq 0)$. The boundary condition above enforces that Yang-Mills theory lives on the d-dimensional boundary at $t = 0$.

This form of the equation is not amenable to perturbation theory, however, due to the presence of nonrenormalizable longitudinal modes in the propagator

$$\tilde{D}^{ab}_{\alpha\beta}(q;t) = g_0^2 \frac{\delta^{ab}}{q^2} \left[\left(\delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2} \right) e^{-q^2 t} + \xi_0 \frac{q_\alpha q_\beta}{q^2} \right]. \tag{5.4}$$

The solution is to add a restoring force which fixes a plane normal to the gauge orbits. For some gauge function F, the covariant derivative provides an appropriate tangent vector. Choosing a Lorenz-like gauge function $F = \partial \cdot B$ as in Sec. 1.5, we have

$$\partial_t B_\mu = D_\nu G_{\nu\mu}[B] + \alpha_0 D_\mu \partial_\nu B_\nu, \qquad B_\mu\big|_{t=0} = A_\mu. \tag{5.5}$$

In Sec. 5.3, we will explore the perturbative solution to the flow equation. Considering only terms linear in B, the leading-order flow equation is simply

$$\partial_t B_\mu = \partial^2 B_\mu + (\alpha_0 - 1)\partial_{\mu\nu} B_\nu, \tag{5.6}$$

and the free bulk field is easily seen to diffuse according to a heat equation. This is most easily analyzed in momentum space with the standard Fourier analysis. Setting $\alpha_0 = 1$ for now, the linearized equation above simplifies to the standard multivariate heat equation, and the fundamental solution is the heat kernel:

$$\tilde{K}(x-y;t) \sim \int_p e^{ip(x-y)} e^{-p^2 t} = (4\pi t)^{-d/2} e^{\frac{-(x-y)^2}{4t}}. \tag{5.7}$$

This may be convoluted with the boundary condition, leading to the free-field solution

$$\tilde{B}_\mu(q;t) = (4\pi t)^{-d/2} \int d^d x\ e^{-iqx} \int d^d y\ e^{-\frac{(x-y)^2}{4t}} A(y) = e^{-q^2 t} \tilde{A}_\mu(q). \tag{5.8}$$

The Gaussian factor represents a delocalization of the gauge field over a d-dimensional sphere with root-mean-squared radius

$$\langle x \rangle_{rms}^2 = \int d^d x\ x^2 e^{-\frac{x^2}{4t}} = 2dt, \tag{5.9}$$

which sets a natural scale $\mu = (2dt)^{-1/2}$ for flowed computations. In practice, it is often simpler to choose $\mu = (2t)^{-1/2} e^{\gamma_E/2}$, corresponding to the $\overline{\text{MS}}$ subtraction point, so that all logarithms vanish in perturbation theory. The Gaussian smearing suppresses the high-energy modes of the boundary field, so that the gauge field is less singular at positive flow time.

The flow equation may be added as a constraint on the action through the use of Lagrange multiplier fields $L_\mu = L_\mu^a t^a$, giving us a new term in the action:

$$S = S_{YM} + S_{gf} + S_{FP} + S_B, \tag{5.10}$$

where

$$S_B = -2 \int dt \int d^d x\ \ \text{Tr}\left\{ L_\mu (\partial_t B_\mu - D_\nu G_{\nu\mu} - \alpha_0 D_\mu \partial_\nu B_\nu) \right\}. \tag{5.11}$$

The coefficient of -2 accounts for the normalization $T_F = -1/2$ of the trace for the particular case of $SU(N)$. Now, the equation of motion for L_μ is exactly the gradient flow equation (5.5). The boundary condition $B_\mu(x;0) = A_\mu(x)$ is implemented by defining

$$B_\mu(x;t) = b_\mu(x,t) + \int d^d y\ K_{\mu\nu}(x-y;t) A_\nu(y), \tag{5.12}$$

where $b_\mu(x;0) = 0$, and $K_{\mu\nu}(x-y;t)$ is a heat kernel which solves linearized flow equation (to be discussed in Sec. 5.3). Since the latter term satisfies Eq. 5.6, the propagator between the L and A fields vanishes, leaving propagators of the form $\langle AA \rangle$, $\langle AB \rangle$, $\langle BB \rangle$, and $\langle LB \rangle$.

5.2 The Fermion Flow

The treatment of fermions in the flowed formalism differs slightly from that of the gauge fields. Since the Dirac action breaks chiral symmetry and is only first order in the spatial derivative, it is not suitable for a gradient flow. Instead, we may construct a covariant flow equation for fermions with the gauge-covariant Laplacian and its adjoint:

$$\Delta = D_\mu D_\mu, \qquad \overleftarrow{\Delta} = \overleftarrow{D}_\mu \overleftarrow{D}_\mu, \tag{5.13}$$

with

$$D_\mu = \partial_\mu + B_\mu, \qquad \overleftarrow{D}_\mu = \overleftarrow{\partial}_\mu - B_\mu, \tag{5.14}$$

Introducing flowed fermion fields χ and $\overline{\chi}$, we define the fermion flow equation

$$\partial_t \chi = \Delta \chi - \alpha_0 \partial_\mu B_\mu \chi, \qquad \chi\big|_{t=0} = \psi, \tag{5.15a}$$

and its adjoint

$$\overline{\chi}\,\overleftarrow{\partial}_t = \overline{\chi}\,\overleftarrow{\Delta} + \alpha_0 \overline{\chi} \partial_\mu B_\mu, \qquad \overline{\chi}\big|_{t=0} = \overline{\psi}. \tag{5.15b}$$

Since the covariant Laplacian coincides with the ordinary Laplacian at leading order, this prescription guarantees that the relaxation of the flowed fermions follows a heat equation as well.

Again, we constrain the action with some (Grassmann-odd) Lagrange multipliers λ and $\overline{\lambda}$, introducing another term to the action:

$$S_{\overline{\chi}\chi} = \int dt \int d^d x \, \left\{ \overline{\lambda}(\partial_t - \Delta + \alpha_0 \partial_\mu B_\mu)\chi + \overline{\chi}(\overleftarrow{\partial}_t - \overleftarrow{\Delta} - \alpha_0 \partial_\mu B_\mu)\lambda \right\} \tag{5.16}$$

with decompositions similar to Eq. 5.12 to enforce the boundary conditions.

5.3 Perturbation Theory

The flow equations, Eq. 5.5 and Eq. 5.15, constitute a system of coupled, nonlinear, parabolic PDEs for the flowed fields, so they are not soluble in any straightforward manner. On the other hand, at leading order in the coupling each reduces to a heat equation, which is readily integrated as in Sec. 5.1. The nonlinear terms may then be treated as perturbations. In the case of the gauge field, the flow equation may be written

$$\partial_t B_\mu = \partial^2 B_\mu + (\alpha_0 - 1)\partial_{\mu\nu} B_\nu + R_\mu, \tag{5.17}$$

where

$$R_\mu = 2[B_\nu, \partial_\nu B_\mu] - [B_\nu, \partial_\mu B_\nu] + (\alpha_0 - 1)[B_\mu, \partial_\nu B_\nu] + [B_\nu, [B_\nu, B_\mu]] \tag{5.18}$$

is the nonlinear remainder that generates radiative corrections to the free solution. The kernel may be easily determined in momentum space:

$$\tilde{K}_{\mu\nu}(q;t) = \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) e^{-q^2 t} + \frac{q_\mu q_\nu}{q^2} e^{-\alpha_0 q^2 t}, \tag{5.19}$$

leading to the solution

$$\tilde{B}_\mu(q;t) = \tilde{K}_{\mu\nu}(q;t)\tilde{A}_\nu(q) + \int_0^t ds\ \tilde{K}_{\mu\nu}(q;t-s)\tilde{R}_\nu(q;s). \tag{5.20}$$

We immediately find the flowed gauge field propagator:

$$\langle \tilde{B}^b_\nu(-q;s)\tilde{B}^a_\mu(q;t)\rangle^{(0)} = g_0^2\frac{\delta^{ab}}{q^2}\left[\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)e^{-q^2(t+s)} + \xi_0\frac{q_\mu q_\nu}{q^2}e^{-\alpha_0 q^2(t+s)}\right], \tag{5.21}$$

which includes the propagators $\langle AA\rangle$ and $\langle AB\rangle$ in the limits of vanishing t and s. The $\langle LB\rangle$ propagator is obtained by considering the Schwinger-Dyson equation for L:

$$\left\langle L^b_\nu(y;s)[\delta_{\mu\rho}\partial_t - \delta_{\mu\rho}\partial^2 - (\alpha_0 - 1)\partial_\mu\partial_\rho]B^a_\rho(x;t)\right\rangle = \delta^{ab}\delta_{\mu\nu}\delta^{(d)}(x-y)\delta(t-s), \tag{5.22}$$

with the condition that $\langle LB\rangle\big|_{s>t=0} = 0$. This has the unique solution

$$\langle L^b_\nu(y;s)B^a_\mu(x;t)\rangle^{(0)} = \int_q e^{iq(x-y)}\delta^{ab}\theta(t-s)\tilde{K}_{\mu\nu}(q;t-s), \tag{5.23}$$

called a (gauge boson) flow or kernel line.

The remainder contains terms quadratic and cubic in the bulk fields which correspond to new three- and four-point vertices. Writing

$$\begin{aligned}\tilde{R}^a_\mu(q;t) =& \frac{1}{2!}\int_{p_1,p_2}(2\pi)^d\delta^{(d)}(q+p_1+p_2)X^{(2,0)}(q,p_1,p_2)^{ab_1b_2}_{\mu\nu_1\nu_2}\tilde{B}^{b_1}_{\nu_1}(-p_1;s)\tilde{B}^{b_2}_{\nu_2}(-p_2;s)\\ &+\frac{1}{3!}\int_{p_1,p_2,p_3}(2\pi)^d\delta^{(d)}(q+p_1+p_2+p_3)\\ &\qquad\times X^{(3,0)}(q,p_1,p_2,p_3)^{ab_1b_2b_3}_{\mu\nu_1\nu_2\nu_3}\tilde{B}^{b_1}_{\nu_1}(-p_1;s)\tilde{B}^{b_2}_{\nu_2}(-p_2;s)\tilde{B}^{b_3}_{\nu_3}(-p_3;s),\end{aligned} \tag{5.24}$$

they are, respectively,

$$X^{(2,0)}(p,q,r)^{abc}_{\mu\nu\rho} = if^{abc}\left\{(r-q)_\mu\delta_{\nu\rho} + 2q_\rho\delta_{\mu\nu} - 2r_\nu\delta_{\rho\mu} + (\alpha_0-1)(q_\nu\delta\rho\mu - r_\rho\delta_{\mu\nu})\right\} \tag{5.25}$$

and

$$\begin{aligned}X^{(3,0)}(p,q,r,s)^{abcd}_{\mu\nu\rho\sigma} =& f^{abe}f^{cde}(\delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\sigma\nu}) + f^{ade}f^{bce}(\delta_{\mu\rho}\delta_{\sigma\nu} - \delta_{\mu\nu}\delta_{\rho\sigma})\\ &+ f^{ace}f^{dbe}(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}).\end{aligned} \tag{5.26}$$

Inspecting the remainder term in S_B, it is obvious that these correspond to B^2L and B^3L vertices. As such, the kernel lines may only connect bulk gauge fields to the flow vertices $X^{(2,0)}$ and $X^{(3,0)}$. One subtlety of the flow lines is that they may not form closed loops. Of course, these cannot appear in the perturbative expansion, but they are allowed when naïvely constructing all graphs, so one must take care to remove these diagrams manually in automated implementations [].

The fermion flow is linearized analogously to the gauge fields:

$$\partial_t \chi = (\partial^2 + \Delta')\chi, \tag{5.27}$$

where $\Delta' = 2B_\mu \partial_\mu - (\alpha_0 - 1)\partial_\mu B_\mu + B_\mu B_\mu$. The leading-order equation is identical to the ordinary heat equation, so the fermion kernel is strictly Gaussian,

$$\tilde{J}(p;t) = e^{-p^2 t}, \tag{5.28}$$

and we have a general solution:

$$\tilde{\chi}(p;t) = \tilde{J}(p;t)\tilde{\psi}(p) + \int_0^t ds\ \tilde{J}(p;t-s)\tilde{\Delta}'\tilde{\chi}(p;s). \tag{5.29}$$

The adjoint flow is similar:

$$\tilde{\bar{\chi}}(p;t) = \tilde{\bar{\psi}}(p)\tilde{\bar{J}}(p;t) + \int_0^t ds\ \tilde{\bar{\chi}}(p;s)\tilde{\overleftarrow{\Delta}}'\tilde{\bar{J}}(p;t-s), \tag{5.30}$$

where $\overleftarrow{\Delta}' = -2B_\mu \partial_\mu + (\alpha_0 - 1)\partial_\mu B_\mu + B_\mu B_\mu$, and $\tilde{\bar{J}}(p;t) = e^{-p^2 t}$. The propagators are obtained exactly as before, leading to

$$\langle \tilde{\chi}(-p;s)\tilde{\bar{\chi}}(p;t)\rangle^{(0)} = \frac{-i\not{p} + m}{p^2 + m + 2} e^{-p^2(t+s)}, \tag{5.31}$$

$$\langle \tilde{\chi}(-p;t)\tilde{\bar{\lambda}}(p;s)\rangle^{(0)} = \theta(t-s)\tilde{J}(p;t-s) \tag{5.32}$$

$$\langle \tilde{\lambda}(-p;s)\tilde{\bar{\chi}}(p;t)\rangle^{(0)} = \theta(t-s)\tilde{\bar{J}}(p;t-s). \tag{5.33}$$

Rewriting the remainder as before,

$$\begin{aligned}
\tilde{\Delta}'\tilde{\chi}(q;t) =& \frac{1}{1!}\int_{p_1,p_2} (2\pi)^d \delta^{(d)}(q + p_1 + p_2) Y^{(1,1)}(q,p_1,p_2)^{a_1}_{\mu_1} \tilde{B}^{a_1}_{\mu_1}(-p_1;s)\tilde{\chi}(-p_2;s) \\
&+ \frac{1}{2!}\int_{p_1,p_2,,p_3} (2\pi)^d \delta^{(d)}(q + p_1 + p_2 + p_3) \\
&\qquad \times Y^{(1,2)}(q,p_1,p_2,p_3)^{a_1 a_2}_{\mu_1 \mu_2} \tilde{B}^{a_1}_{\mu_1}(-p_1;s)\tilde{B}^{a_2}_{\mu_2}(-p_2;s)\tilde{\chi}(-p_3;s),
\end{aligned} \tag{5.34}$$

we find two more vertices,

$$Y^{(1,1)}(p,q,r)^a_\mu = -it^a \left\{(1 - \alpha_0)r_\mu + 2q_\mu\right\} \tag{5.35}$$

and

$$Y^{(1,2)}(p,q,r,s)^{ab}_{\mu\nu} = \delta_{\mu\nu}\left\{t^a, t^b\right\}, \tag{5.36}$$

corresponding to $\chi\bar{\lambda}B$ and $\chi\bar{\lambda}B^2$. The adjoint flow equation, too, generates two vertices;

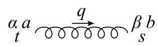
$$\overline{Y}^{(1,1)}(p,q,r)^a_\mu = it^a \left\{(1 - \alpha_0)r_\mu + 2q_\mu\right\} \tag{5.37}$$

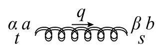
and

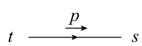
$$\overline{Y}^{(1,2)}(p,q,r,s)^{ab}_{\mu\nu} = \delta_{\mu\nu}\left\{t^a, t^b\right\}, \tag{5.38}$$

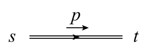
corresponding to $\lambda\overline{\chi}B$ and $\lambda\overline{\chi}B^2$.

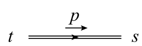
Altogether, we have the following Feynman rules. The flowed propagators are

$\langle\tilde{B}\tilde{B}\rangle$: 
$$= g_0^2\frac{\delta^{ab}}{q^2}\left[\left(\delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}\right)e^{-q^2(t+s)} + \xi_0\frac{q_\alpha q_\beta}{q^2}e^{-\alpha_0 q^2(t+s)}\right], \tag{5.39a}$$

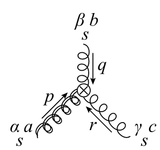
$\langle\tilde{L}\tilde{B}\rangle$: 
$$= \delta^{ab}\theta(t-s)\left[\left(\delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}\right)e^{-q^2(t-s)} + \frac{q_\alpha q_\beta}{q^2}e^{-\alpha_0 q^2(t-s)}\right], \tag{5.39b}$$

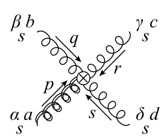
$\langle\tilde{\chi}\tilde{\overline{\chi}}\rangle$: 
$$= \frac{-i\not{p}+m}{p^2+m+2}e^{-p^2(t+s)}, \tag{5.39c}$$

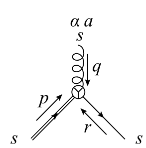
$\langle\tilde{\chi}\tilde{\overline{\lambda}}\rangle$: 
$$= \theta(t-s)e^{-p^2(t-s)}, \tag{5.39d}$$

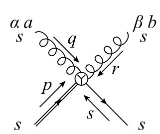
$\langle\tilde{\lambda}\tilde{\overline{\chi}}\rangle$: 
$$= \theta(t-s)e^{-p^2(t-s)}. \tag{5.39e}$$

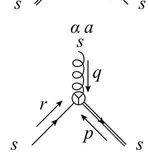
The vertices are

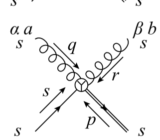
$\langle\tilde{B}^2\tilde{L}\rangle$: 
$$= if^{abc}\int_0^\infty ds\big[\delta_{\beta\gamma}(q-r)_\alpha + 2\delta_{\gamma\alpha}r_\beta - 2\delta_{\alpha\beta}q_\gamma + (\alpha_0-1)(\delta_{\alpha\beta}r_\gamma - \delta_{\gamma\alpha}q_\beta)\big], \tag{5.40a}$$

$\langle\tilde{B}^3\tilde{L}\rangle$: 
$$= -\int_0^\infty ds\big[f^{abe}f^{cde}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\gamma\beta}) + f^{ace}f^{bde}(\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\delta}\delta_{\gamma\beta}) + f^{ade}f^{bce}(\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\gamma}\delta_{\beta\delta})\big], \tag{5.40b}$$

$\langle\tilde{\chi}\tilde{B}\tilde{\overline{\lambda}}\rangle$: 
$$= -it^a\int_0^\infty ds\left[2r_\alpha + (1-\alpha_0)q_\alpha\right], \tag{5.40c}$$

$\langle\tilde{\chi}\tilde{B}^2\tilde{\overline{\lambda}}\rangle$: 
$$= \delta_{\alpha\beta}\{t^a, t^b\}\int_0^\infty ds, \tag{5.40d}$$

$\langle\tilde{\lambda}\tilde{B}\tilde{\overline{\chi}}\rangle$: 
$$= it^a\int_0^\infty ds\left[2r_\alpha + (1-\alpha_0)q_\alpha\right], \tag{5.40e}$$

$\langle\tilde{\lambda}\tilde{B}^2\tilde{\overline{\chi}}\rangle$: 
$$= \delta_{\alpha\beta}\{t^a, t^b\}\int_0^\infty ds. \tag{5.40f}$$

Above, fermions are represented by oriented solid lines; gauge bosons by curly lines; fermionic flow lines by oriented, double solid lines; and gauge boson flow lines by double curly lines. This notation differs notably from much of the literature, wherein all flow lines are single straight lines with an adjacent arrow indicating the direction of increasing flow time, determined by the attached Heaviside θ functions. In order to avoid a proliferation of arrows, we choose the double-line notation. The direction of flow time is unambiguous, since all subgraphs consisting of only flow lines and vertices are directed trees with each child vertex at a flow time less than or equal than that of its parent and the root at the maximum flow time (flow-line loops having been already excluded).

The integrals over s in the vertices are meant to be performed only after all attached legs are taken into the integrand. The flow vertices are inscribed by an X or Y to signify bosonic or fermionic vertices. Note that flow lines cannot be cut when constructing 1PI diagrams, since they represent genuine corrections to the flowed field.

Chapter 6

Renormalization and BRST Symmetry

A remarkable feature of the Yang-Mills gradient flow is that once the boundary theory is renormalized, the bulk gauge fields are finite to all orders. This is not the case for bulk fermions, though they may be rendered finite by a multiplicative field strength renormalization. in order to see this, we will evaluate the one-loop propagators of both the bulk gauge field and the bulk fermions. We fix $\alpha_0 = 1$ but leave ξ_0 free, since it requires renormalization at one-loop. To perform the momentum integrals, we use dimensional regularization with $d = 4 - 2\epsilon$ and employ the novel method introduced in App. **??**

6.1 Gauge Field Self-Energy

In Chapter 2, we showed that the bare gluon propagator,

$$\begin{aligned}\langle \tilde{A}^b_\beta(-q)\tilde{A}^a_\alpha(q)\rangle_0 = g_0^2\frac{\delta^{ab}}{q^2}\left[\Pi_{\alpha\beta}+\xi_0\Lambda_{\alpha\beta}\right] - \frac{g_0^4}{(4\pi)^2}\frac{\delta^{ab}}{q^2}\Bigg[&\left(\frac{13-3\xi_0}{6}T_A+\frac{2}{3}n_f\right)L_0 \\ &+\frac{1}{4}\left(\xi_0^2+2\xi_0+\frac{97}{9}\right)T_A+\frac{10}{9}n_f\Bigg]\Pi_{\alpha\beta}+\mathcal{O}(g_0^6),\end{aligned} \tag{6.1}$$

may be renormalized by making the replacements

$$A_0 = Z_g Z_\xi A, \qquad g_0 = \mu^\epsilon Z_g g, \qquad \xi_0 = Z_\xi \xi. \tag{6.2}$$

At one-loop order, the propagator of the bulk gauge fields, $\langle \tilde{B}^b_\beta(-q)\tilde{B}^a_\alpha(q)\rangle$ is the sum of eight diagrams (Fig. 6.1). The first four of these diagrams ((a)-(d)) are identical to the unflowed diagrams up to the external fields. Since the difference is $\mathcal{O}(t)$, both the divergent and finite parts are unaffected at small t. The four additional diagrams ((e)-(h)) contain flow lines and vertices, representing the nonlinear terms in the flow equation. These are the first diagrams which exhibit the incomplete gamma integrals of App. **??**, due to Gaussian factors within the loops. For demonstration, we will calculate diagram (d) explicitly. For brevity, we will set $\xi_0 = \alpha_0 = 1$, writing the full result only at the end.

Starting from the Feynman rules, Eqs. 5.39 and Eqs. 5.40, we have (dropping external

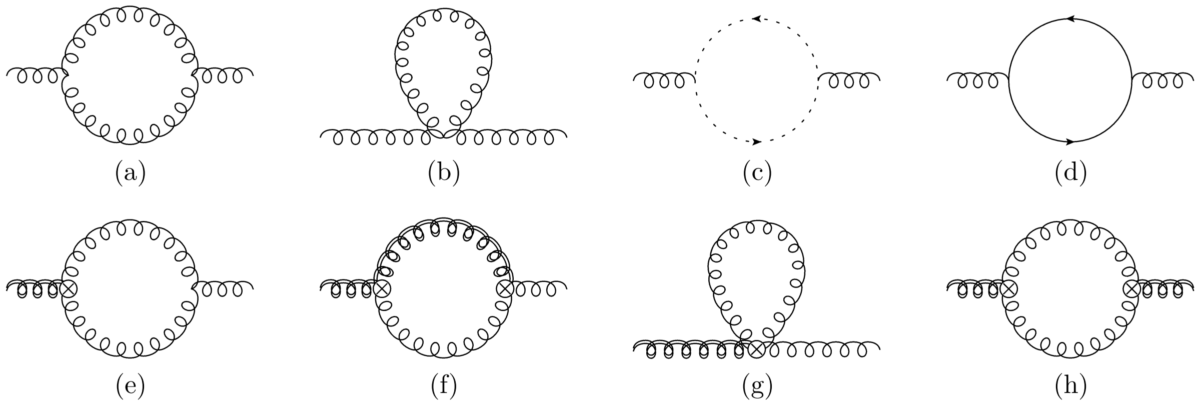


Figure 6.1: One-loop contributions to the propagator of the flowed gauge field

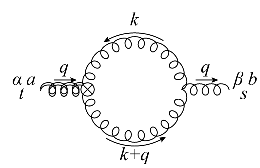


Figure 6.2: Diagram 6.1e with labels

indices and the outgoing AA leg)

$$\begin{aligned}\Gamma_e = \int_k \frac{if^{bcd}}{g_0^2}&\left[-(q+k)_\delta\delta_{\beta\gamma} + (2k-q)_\beta\delta_{\gamma\delta} + (2q-k)_\gamma\delta_{\delta\beta}\right]\\ &\times if^{adc}\left[\delta_{\delta\gamma}(2k-q)_\alpha - 2\delta_{\gamma\alpha}k_\delta + 2\delta_{\alpha\delta}(q-k)_\gamma\right]\\ &\times \int_0^t du\ \left(g_0^2\frac{e^{-k^2u}}{k^2}\right)\left(g_0^2\frac{e^{-(q-k)^2u}}{(q-k)^2}\right)e^{-q^2(t-u)}.\end{aligned} \tag{6.3}$$

Simplifying the numerical and color factors, collecting like terms in k, and writing the $(q-k)$ propagator in Schwinger parameters,

$$\begin{aligned}\Gamma_e = -g_0^2 T_A \int_0^t du \int_0^\infty dz\ e^{-q^2(z+t)} \int_k \frac{e^{-k^2(2u+z)}}{k^2} e^{2(k\cdot q)(u+z)}&\\ \times\Big\{\left[(d-5)q_\alpha q_\beta + 4q^2\delta_{\alpha\beta}\right] - 2\left[(d-2)(q_\alpha\delta_{\beta\mu} + q_\beta\delta_{\alpha\mu}) + 2q_\mu\delta_{\alpha\beta}\right]k_\mu&\\ + 2\left[(d-2)(\delta_{\alpha\nu}\delta_{\beta\mu} + \delta_{\beta\nu}\delta_{\alpha\mu}) + 2\delta_{\mu\nu}\delta_{\alpha\beta}\right]k_\mu k_\nu\Big\},&\end{aligned} \tag{6.4}$$

it is clear that the only term with angular dependence is $e^{2(k\cdot q)(u+z)}$. If we expand this as a MacLaurin series, we can again collect like powers of the loop momentum and discard all odd powers due to the symmetry of the integral. The first bracketed term above is constant with respect to k, so it multiplies only even powers of $k\cdot q$, and we can reindex $n \to 2n$. Likewise, the second and third terms are respectively odd and even in k, so they are reindexed according to $n \to 2n+1$ and $n \to 2n$:

$$\begin{aligned}\Gamma_e = -g_0^2 T_A \int_0^t du \int_0^\infty dz\ e^{-q^2(z+t)} \int_k \frac{e^{-k^2(2u+z)}}{k^2} \sum_{n=0}^{\infty} \frac{(2(u+z))^{2n}}{(2n)!} q_{I_{2n}} k_{I_{2n}} \\ \times \Big\{ \left[(d-5)q_\alpha q_\beta + 4q^2\delta_{\alpha\beta}\right] + 2\left[(d-2)(\delta_{\alpha\nu}\delta_{\beta\mu} + \delta_{\beta\nu}\delta_{\alpha\mu}) + 2\delta_{\mu\nu}\delta_{\alpha\beta}\right] k_\mu k_\nu \\ - \frac{4(u+z)}{2n+1}\left[(d-2)(q_\alpha\delta_{\beta\mu} + q_\beta\delta_{\alpha\mu}) + 2q_\mu\delta_{\alpha\beta}\right] q_{\mu_{2n+1}} k_{\mu_{2n+1}} k_\mu \Big\},\end{aligned} \tag{6.5}$$

All terms are now even in k, so they may be decomposed according to Eq. **??** and replaced by:

$$q_{I_{2n}} k_{I_{2n}} \to \frac{1}{(d)_{n,2}} q_{I_{2n}} S^{(2n)}_{I_{2n}} = \frac{(2n-1)!!}{(d)_{n,2}} \left(q^2\right)^n \left(k^2\right)^n, \tag{6.6a}$$

$$q_{I_{2n+1}} k_{I_{2n+1}\mu} \to \frac{1}{(d)_{n+1,2}} q_{I_{2n+1}} S^{(2n+2)}_{I_{2n+1}\mu} = \frac{(2n+1)!!}{(d)_{n+1,2}} \left(q^2\right)^n \left(k^2\right)^{n+1} q_\mu, \tag{6.6b}$$

$$q_{I_{2n}} k_{I_{2n}\mu\nu} \to \frac{1}{(d)_{n+1,2}} q_{I_{2n}} S^{(2n+2)}_{I_{2n}\mu\nu} = \frac{(2n-1)!!}{(d)_{n+1,2}} \left(q^2\right)^n \left(k^2\right)^{n+1} \left(\delta_{\mu\nu} + 2n\frac{q_\mu q_\nu}{q^2}\right). \tag{6.6c}$$

The momentum integrals are now in the form of Eq. **??**, and we can simplify the expression by making the substitutions $u = t\upsilon$, $z = t\zeta$, and $\tau = q^2 t$:

$$\begin{aligned}\Gamma_e = -g_0^2 \frac{T_A \delta^{ab}}{(4\pi)^2}(4\pi t)^{2-d/2} \int_0^1 d\upsilon \int_0^\infty d\zeta \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \frac{(\zeta+\upsilon)^{2n}}{(\zeta+2\upsilon)^{d/2+n}} e^{-\tau(\zeta+1)} \\ \times \Big\{ \frac{2\tau(\zeta+2\upsilon)}{d+2n-2}\left((d-5)\frac{q_\alpha q_\beta}{q^2} + 4\delta_{\alpha\beta}\right) \\ - \frac{8\tau(\zeta+\upsilon)}{d+2n}\left((d-2)\frac{q_\alpha q_\beta}{q^2} + \delta_{\alpha\beta}\right) \\ + \frac{8}{d+2n}\left((d+n-1)\delta_{\alpha\beta} + (d-2)n\frac{q_\alpha q_\beta}{q^2}\right) \Big\}.\end{aligned} \tag{6.7}$$

There are a few ways to sum and integrate this expression. One can recast the factor $(\zeta+2\upsilon)^{-d/2-n}$ as a binomial series so that the integrals are simpler. Alternatively, integrating in υ first produces hypergeometric functions, and integrating in ζ or summing over n produces incomplete gamma functions. By replacing these special functions by their integral or series definitions, a complete solution can be obtained, but the intermediate expressions are fairly intractable and lead to the same result. Instead, it is far simpler to note that for the first two bracketed terms above are at least $\mathcal{O}(t)$ for all $n \geq 0$, as is the third term for $n \geq 1$, so they may be discarded. We are left with

$$\Gamma_e = -8g_0^2 \frac{T_A}{(4\pi)^2} \frac{d-1}{d}(4\pi t)^{2-d/2} \int_0^1 d\upsilon \int_0^\infty d\zeta\ \frac{e^{-\tau(\zeta+1)}}{(\zeta+2\upsilon)^{d/2}} \delta^{ab}\delta_{\alpha\beta} + \mathcal{O}(\tau). \tag{6.8}$$

Integrating over ζ, we have

$$\Gamma_e = -8g_0^2 \frac{T_A}{(4\pi)^2}\frac{d-1}{d}(4\pi t)^{2-d/2}e^{-\tau}\tau^{d/2-1}\int_0^1 d\upsilon\ e^{2\tau\upsilon}\Gamma\left(1-\frac{d}{2},2\tau\upsilon\right)\delta^{ab}\delta_{\alpha\beta}+\mathcal{O}(\tau). \tag{6.9}$$

Integrating over υ, we are left with

$$\begin{aligned}\Gamma_e = -8g_0^2 \frac{T_A}{(4\pi)^2}&\frac{d-1}{d(d-2)}(4\pi t)^{2-d/2}\tau^{d/2-2} \\ &\times\left\{e^{-\tau}\Gamma\left(2-\frac{d}{2}\right)-e^{\tau}\Gamma\left(2-\frac{d}{2},2\tau\right)\right\}\delta^{ab}\delta_{\alpha\beta}+\mathcal{O}(\tau),\end{aligned} \tag{6.10}$$

which may be expanded in ϵ and t to zeroth order:

$$\Gamma_e = -3g_0^2\frac{T_A}{(4\pi)^2}\left\{\frac{1}{\epsilon}+\log(8\pi t)+\frac{5}{6}\right\}\delta^{ab}\delta_{\alpha\beta}+\mathcal{O}(\epsilon,t). \tag{6.11}$$

Proceeding as above for generic ξ_0, we find

$$\Gamma_e(q;t) = -\frac{1}{2}\cdot\frac{3}{2}g_0^2\frac{T_A}{(4\pi)^2}(\xi_0+1)\left\{\frac{1}{\epsilon}+\log(8\pi t)+\frac{5}{6}\right\}\delta^{ab}\left(\Lambda_{\alpha\beta}+\Pi_{\alpha\beta}\right)+\mathcal{O}(\epsilon,t), \tag{6.12a}$$

$$\Gamma_f(q;t) = \frac{1}{8}g_0^2\frac{T_A}{(4\pi)^2}\left\{(\xi_0-9)\left[\frac{1}{\epsilon}+\log(8\pi t)\right]+\frac{1}{2}(\xi_0+3)\right\}\delta^{ab}\left(\Lambda_{\alpha\beta}+\Pi_{\alpha\beta}\right)+\mathcal{O}(\epsilon,t), \tag{6.12b}$$

$$\Gamma_g(q;t) = \frac{1}{2}\cdot\frac{3}{4}g_0^2\frac{T_A}{(4\pi)^2}\left\{(\xi_0+3)\left[\frac{1}{\epsilon}+\log(8\pi t)\right]+\frac{1}{6}(5\xi_0+3)\right\}\delta^{ab}\left(\Lambda_{\alpha\beta}+\Pi_{\alpha\beta}\right)+\mathcal{O}(\epsilon,t), \tag{6.12c}$$

$$\Gamma_h(q;t) = \frac{1}{2}\cdot\mathcal{O}(t,s), \tag{6.12d}$$

where the symmetry factors have been written explicitly. There are three additional diagrams which are simply mirror images of (e)-(g), related by the interchange $t\leftrightarrow s$. Summing all contributions with external legs included, the bare propagator is

$$\begin{aligned}\langle\tilde{B}^b_\beta(-q)\tilde{B}^a_\alpha(q)\rangle_0 = g_0^2\frac{\delta^{ab}}{q^2}&\left[\Pi_{\alpha\beta}+\xi_0\Lambda_{\alpha\beta}\right] \\ &-\frac{g_0^4}{(4\pi)^2}\frac{\delta^{ab}}{q^2}\Bigg\{\Bigg[\left(\frac{13-3\xi_0}{6}T_A+\frac{2}{3}n_f\right)\left(\frac{1}{\epsilon}+\log\left(\frac{4\pi}{q^2}\right)-\gamma_E\right) \\ &\qquad+\frac{\xi_0+3}{4}T_A\left(\frac{2}{\epsilon}+\log(8\pi t)+\log(8\pi s)\right) \\ &\qquad+\frac{9\xi_0(\xi_0+4)+115}{36}T_A+\frac{10}{9}n_f\Bigg]\Pi_{\alpha\beta} \\ &\qquad+\Bigg[\frac{\xi_0(\xi_0+3)}{4}T_A\left(\frac{2}{\epsilon}+\log(8\pi t)+\log(8\pi s)\right) \\ &\qquad\quad+\frac{\xi_0(\xi_0+1)}{2}T_A\Bigg]\Lambda_{\alpha\beta}+\mathcal{O}(\epsilon,t,s)\Bigg\} \\ &+\mathcal{O}(g_0^6).\end{aligned} \tag{6.13}$$

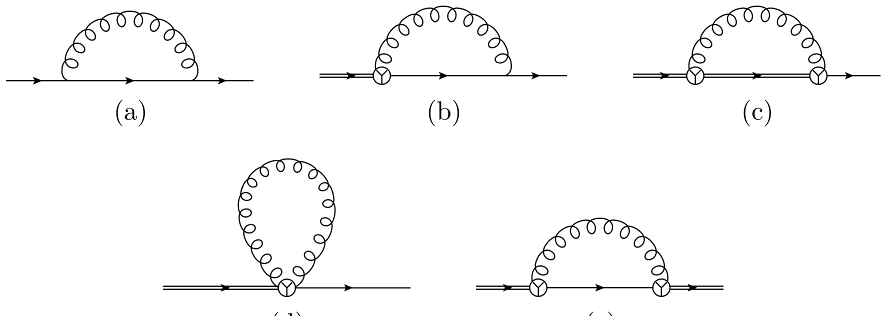


Figure 6.3: One-loop contributions to the propagator of the flowed fermion fields

Replacing the bare coupling g_0 and gauge-fixing parameter ξ_0 by their renormalized counterparts as in Eq. 6.2, we find a finite result without any field renormalization:

$$
\begin{aligned}
\langle \tilde{B}^b_\beta(-q)\tilde{B}^a_\alpha(q)\rangle = g^2\frac{\delta^{ab}}{q^2}\left[\Pi_{\alpha\beta}+\xi\Lambda_{\alpha\beta}\right] \\
- \frac{g^4}{(4\pi)^2}\frac{\delta^{ab}}{q^2}\Bigg\{\Bigg[\left(\frac{13-3\xi}{6}T_A+\frac{2}{3}n_f\right)\left(\log\left(\frac{4\pi\mu^2}{q^2}\right)-\gamma_E\right) \\
+\frac{\xi+3}{4}T_A\left(\log\left(8\pi\mu^2 t\right)+\log\left(8\pi\mu^2 s\right)\right) \\
+\frac{9\xi(\xi+4)+115}{36}T_A+\frac{10}{9}n_f\Bigg]\Pi_{\alpha\beta} \\
+\Bigg[\frac{\xi(\xi+3)}{4}T_A\left(\log\left(8\pi\mu^2 t\right)+\log\left(8\pi\mu^2 s\right)\right) \\
+\frac{\xi(\xi+1)}{2}T_A\Bigg]\Lambda_{\alpha\beta}+\mathcal{O}(t,s)\Bigg\} \\
+\mathcal{O}(g^6).
\end{aligned}
\tag{6.14}
$$

Then, at least to one-loop order, the bulk gauge fields require no renormalization. In fact, we will show that there are no bulk counterterms for the gauge field at any order in Sec. 6.3.

6.2 Fermion Self-Energy

At one loop, the fermion self-energy receives contribution from eight diagrams with five unique topologies, Fig. 6.3. Since the relevant fermion masses, m_u, m_d, m_s (cf. Ch. **??**), are all far less than than typical hadronic scales, $\Lambda \sim 1$ GeV, and since they are identically zero in the chiral limit, we consider them perturbations and keep only the leading order. Treating the integrals as we did in evaluating the gauge field propagator (*e.g.*, Fig. **??**), we calculate:

$$\Gamma_a(p;t) = -g_0^2 \frac{C_2(F)}{(4\pi)^2}\frac{1}{p^2}\Bigg\{\xi_0\left[\frac{1}{\epsilon}+\log\left(\frac{4\pi}{p^2}\right)-\gamma_E+1\right] i\not{p}$$
$$+\left[(3-\xi_0)\left(\frac{1}{\epsilon}+\log\left(\frac{4\pi}{p^2}\right)-\gamma_E\right)+4\right] m_0\Bigg\} + \mathcal{O}(\epsilon,t,s,m_0^2), \tag{6.15a}$$

$$\Gamma_b(q;t) = -g_0^2 \frac{C_2(F)}{(4\pi)^2}\frac{\xi_0}{p^2}\left(\frac{1}{\epsilon}+\log(8\pi t)+1\right)\left(-i\not{p}+m_0\right)+\mathcal{O}(\epsilon,t,s,m_0^2), \tag{6.15b}$$

$$\Gamma_c(q;t) = \mathcal{O}(t,s), \tag{6.15c}$$

$$\Gamma_d(q;t) = \frac{1}{2}\cdot g_0^2 \frac{C_2(F)}{(4\pi)^2}\frac{1}{p^2}\left[(\xi_0+3)\left(\frac{1}{\epsilon}+\log(8\pi t)\right)+\xi_0+1\right]\left(-i\not{p}+m_0\right)+\mathcal{O}(\epsilon,t,s,m_0^2), \tag{6.15d}$$

$$\Gamma_e(q;t) = \mathcal{O}(t,s). \tag{6.15e}$$

Summing these along with three additional diagrams related to (b)-(d) by the interchange $t \leftrightarrow s$, the total bare propagator is

$$\begin{aligned}
&\langle\tilde{\chi}(-p;s)\tilde{\bar{\chi}}(p;t)\rangle_0 \\
&\quad = \frac{-i\not{p}+m_0}{p^2} \\
&\quad + g_0^2\frac{C_2(F)}{(4\pi)^2}\frac{1}{p^2}\Bigg\{-i\not{p}\left[\frac{3}{\epsilon}+\xi_0\log\left(\frac{4\pi}{\gamma' p^2}\right)+\frac{3-\xi_0}{2}\left(\log(8\pi t)+\log(8\pi s)\right)+1\right] \\
&\qquad + m_0\left[(\xi_0-3)\left(\log\left(\frac{4\pi}{\gamma' p^2}\right)-\frac{1}{2}\log(8\pi t)-\frac{1}{2}\log(8\pi s)\right)-\xi_0-3\right] \\
&\qquad + \mathcal{O}(\epsilon)\Bigg\} \\
&\quad + \mathcal{O}(g_0^4, m_0^2, t, s).
\end{aligned} \tag{6.16}$$

Replacing the bare mass, coupling, and gauge-fixing parameter with renormalized parameters,

$$m_0 = Z_m m, \qquad g_0 = \mu^\epsilon Z_g g, \qquad \xi_0 = Z_\xi \xi, \tag{6.17}$$

the bare propagator becomes

$$
\begin{aligned}
&\langle\tilde{\chi}(-p;s)\tilde{\overline{\chi}}(p;t)\rangle_0 \\
&\quad=\frac{-i\not{p}+m}{p^2} \\
&\qquad+g^2\frac{C_2(F)}{(4\pi)^2}\frac{1}{p^2}\Bigg\{-i\not{p}\left[\frac{3}{\epsilon}+\xi\log\left(\frac{4\pi\mu^2}{\gamma' p^2}\right)+\frac{3-\xi}{2}\left(\log\Big(8\pi\mu^2 t\Big)+\log\Big(8\pi\mu^2 s\Big)\right)+1\right] \\
&\qquad\qquad+m\left[\frac{3}{\epsilon}+(\xi-3)\left(\log\left(\frac{4\pi\mu^2}{\gamma' p^2}\right)-\frac{1}{2}\log\Big(8\pi\mu^2 t\Big)-\frac{1}{2}\log\Big(8\pi\mu^2 s\Big)\right)-\xi-3\right] \\
&\qquad\qquad+\mathcal{O}(\epsilon)\Bigg\} \\
&\qquad+\mathcal{O}(g^4,m^2,t,s),
\end{aligned}
\tag{6.18}
$$

and there is an overall pole of $3/\epsilon$ remaining in both the mass and kinetic terms. This may be canceled by defining a renormalized bulk fermion field:

$$\chi_0=Z_\chi^{1/2}\chi, \qquad \overline{\chi}_0=\overline{\chi}Z_\chi^{1/2}, \tag{6.19}$$

where

$$Z_\chi=1+g^2\frac{C_2(F)}{(4\pi)^2}\frac{3}{\epsilon}+\mathcal{O}(g^4). \tag{6.20}$$

The fully renormalized one-loop propagator is thus

$$
\begin{aligned}
&\langle\tilde{\chi}(-p;s)\tilde{\overline{\chi}}(p;t)\rangle \\
&\quad=\frac{-i\not{p}+m}{p^2} \\
&\qquad+g^2\frac{C_2(F)}{(4\pi)^2}\frac{1}{p^2}\Bigg\{-i\not{p}\left[\xi\log\left(\frac{4\pi\mu^2}{\gamma' p^2}\right)+\frac{3-\xi}{2}\left(\log\Big(8\pi\mu^2 t\Big)+\log\Big(8\pi\mu^2 s\Big)\right)+1\right] \\
&\qquad\qquad+m\left[(\xi-3)\left(\log\left(\frac{4\pi\mu^2}{\gamma' p^2}\right)-\frac{1}{2}\log\Big(8\pi\mu^2 t\Big)-\frac{1}{2}\log\Big(8\pi\mu^2 s\Big)\right)-\xi-3\right]\Bigg\} \\
&\qquad+\mathcal{O}(g^4,m^2,t,s).
\end{aligned}
\tag{6.21}
$$

In the flowed action, the only fermionic counterterm allowed by gauge invariance, Grassmann parity, and the counting of engineering dimensions,

$$[\chi]=\frac{d-1}{2}, \qquad [\lambda]=\frac{d+1}{2}, \tag{6.22}$$

is proportional to

$$\int dt\int d^dx\ \left(\overline{\chi}\lambda+\overline{\lambda}\chi\right). \tag{6.23}$$

Note, however, that since the integrand must generate at least one flowed propagator at every order in perturbation theory, it is always exponentially suppressed, and the integral is necessarily convergent, so there are no bulk fermionic counterterms. This is not the case on the boundary, where the corresponding term,

$$S_{\overline{\lambda},\lambda} = \int d^d x \ \left(\overline{\psi}\lambda|_{t=0} + \overline{\lambda}|_{t=0}\psi\right), \tag{6.24}$$

is required by BRST invariance. This forces us to reciprocally renormalize the fermionic Lagrange multipliers:

$$\lambda_0 = Z_\chi^{-1/2}\lambda, \qquad \overline{\lambda}_0 = \overline{\lambda} Z_\chi^{-1/2}. \tag{6.25}$$

We will return to this at the end of Sec. 6.4.

6.3 BRST Symmerty in $(d+1)$ Dimensions

The flow equation, Eq. 5.5, is invariant under a gauge transformation, Eq. 1.7, so long as the gauge function ω satisfies

$$\left(\partial_t - \alpha_0 D_\mu \partial_\mu\right)\omega = 0. \tag{6.26}$$

This condition may be fixed in a manner similar to the Faddeev-Popov construction, namely, by introducing a bulk ghost field d and a bulk antighost $\overline{d}$ with the action

$$S_{\overline{d}d} = -2\int dt \int d^d x \ \mathrm{Tr}\left\{\overline{d}(\partial_t - \alpha_0 D_\mu\partial_\mu)d\right\}. \tag{6.27}$$

The ghost field has the boundary condition $d_{t=0} = c$, while the antighost is left unfixed on the boundary, since it acts as a Lagrange multiplier generating a flow equation for d. The bulk ghost field then receives perturbative corrections just as the gauge and fermion fields do. In particular, the ghost field flow equation has the recursive solution

$$\begin{aligned}\tilde{d}(p;t) &= e^{-\alpha_0 p^2 t}\tilde{c}(p) \\ &\quad + \int_0^t ds\ e^{-\alpha_0 p^2 (t-s)} \int_{p_1,p_2} (2\pi)^d \delta^{(d)}(p+p_1+p_2) \\ &\qquad \times X^{(1,1)}(p,p_1,p_2)^{aa_1a_2}_{\mu_1} \tilde{B}^{a_1}_{\mu_1}(-p_1;s)\tilde{d}^{a_2}(-p_2;s),\end{aligned} \tag{6.28}$$

which gives us the propagators

$$\langle \tilde{d}^b(-p;s)\tilde{\overline{d}}^a(p;t)\rangle^{(0)} = \delta^{ab}\theta(t-s)e^{-\alpha_0 p^2 t} \tag{6.29}$$

and, by virtue of the boundary condition,

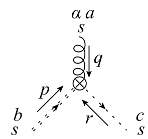
$$\langle \tilde{d}^b(-p;s)\tilde{\overline{c}}^a(p;t)\rangle^{(0)} = g_0^2\delta^{ab}\frac{e^{-\alpha_0 p^2 t}}{p^2}. \tag{6.30}$$

There is also a vertex

$$X^{(1,1)}(p,q,r)^{abc}_\mu = -i\alpha_0 f^{abc} r_\mu, \tag{6.31}$$

giving us the Feynman rule

$\langle \tilde{d}\tilde{B}\tilde{\overline{d}} \rangle$:



$$= i\alpha_0 f^{abc} r_\alpha, \tag{6.32}$$

Where $\overline{d}$ is represented by a double dotted line, d by the standard dotted line. These fields generally do not enter perturbation theory, but they are necessary for a complete BRST-invariant action:

$$S = S_d + S_{d+1} \tag{6.33}$$

where the unflowed action is the sum

$$S_d = S_{YM} + S_D + S_{FP} + S_{gf}, \tag{6.34}$$

and the flowed part of the action is

$$S_{d+1} = S_B + S_{\overline{\chi}\chi} + S_{\overline{d}d}. \tag{6.35}$$

The boundary theory, S_d, was shown to be invariant under BRST transformations in Sec. 1.5. Extending that procedure to the flowed theory, the variations of the bulk gauge, fermion, and ghost fields are exactly like those at $t = 0$, Eqs. 1.55-1.56,

$$\delta\chi = -\theta d\chi, \quad \delta\overline{\chi} = -\theta\overline{\chi}d, \quad \delta B_\mu = \theta D_\mu d, \quad \delta d_\mu = -\theta d^2. \tag{6.36}$$

For each of these, the associated Lagrange multipliers transform similarly,

$$\delta\lambda = -\theta d\lambda, \quad \delta\overline{\lambda} = -\theta\overline{\chi}d, \quad \delta L_\mu = \theta[L_\mu, d], \tag{6.37}$$

with the exception of the bulk antighost field, whose variation has an unusual structure:

$$\delta\overline{d} = \theta\left\{D_\mu L_\mu - \{d, \overline{d}\} + \overline{\lambda}t^a\chi t^a - \overline{\chi}t^a\lambda t^a\right\}. \tag{6.38}$$

This last expression is derivable by extending the configuration space to include components of the gauge field in the t-direction, $B = (B_\mu, B_t)$, so that gauge transformations assume a $(d+1)$-dimensional form []. Under these variations, the total action is invariant:

$$\delta S = 0. \tag{6.39}$$

6.4 Perturbative Renormalizability

In order to prove the renormalizability, we follow Ref. [], omitting many details. The Slavnov-Taylor (Ward) identity associated to the BRST symmetry of the flowed theory is the Zinn-Justin (ZJ) equation [], which requires a few definitions in advance. First we introduce a source J for each field:

$$S_J = \sum_\phi \int d^dx\ (\pm J_\phi \phi) + \sum_\Phi \int d^dx\ \int dt\ (\pm J_\Phi \Phi) \tag{6.40}$$

(where the sums are taken over all boundary fields ϕ and all bulk fields Φ with traces implied where necessary and signs accounting for canonical ordering of anticommuting fields), as well as a source K for each variation:

$$S_K = \sum_{\phi} \int d^d x \ (\pm K_{\phi} \delta\phi) + \sum_{\Phi} \int d^d x \ \int dt \ (\pm K_{\Phi} \delta\Phi). \tag{6.41}$$

We now define the effective action functional, which produces all 1PI correlation functions, as the Legendre transform of the energy with respect to the sources:

$$\Gamma[K, \phi, \Phi] = -\log Z[J, K] - S_J, \qquad Z[J, K] = \int \mathcal{D}[\phi, \Phi] e^{-S - S_J - S_K}. \tag{6.42}$$

Working in the off-shell scheme, one may simplify the following arguments by eliminating the Nakanishi-Lautrup field B^a and the antighost $\bar{c}^a$ through a shift in the effective action,

$$\tilde{\Gamma}[K, \phi, \Phi] = \Gamma[K, \phi, \Phi] - \frac{1}{T_F} \int d^d x \ \operatorname{Tr} B \frac{\delta \Gamma}{\delta B}, \tag{6.43}$$

which is absorbed by the source of the variation of the gauge field on the boundary, K_A. Now the Zinn-Justin equation assumes the form

$$\sum_{\phi} \int d^d x \ \left(\pm \frac{\delta \tilde{\Gamma}}{\delta \phi} \frac{\delta \tilde{\Gamma}}{\delta K_{\phi}} \right) + \sum_{\Phi} \int d^d x \ \int dt \ \left(\pm \frac{\delta \tilde{\Gamma}}{\delta \Phi} \frac{\delta \tilde{\Gamma}}{\delta K_{\Phi}} \right) = 0 \tag{6.44}$$

Renormalizability is then proven by induction on n, the order of the perturbative expansion of the effective action:

$$\tilde{\Gamma} = \sum_{n=0}^{\infty} g^{2n} \tilde{\Gamma}^{(n)} \tag{6.45}$$

where $\tilde{\Gamma}^{(0)} = S + S_K$ with all counterterms set to zero. Defining now the BRST operator in the form of a functional derivative,

$$\mathcal{D}_n = \sum_{\phi} \int d^d x \ \left(\frac{\delta \tilde{\Gamma}^{(n)}}{\delta \phi} \frac{\delta}{\delta K_{\phi}} + \frac{\delta \tilde{\Gamma}^{(n)}}{\delta K_{\phi}} \frac{\delta}{\delta \phi} \right) + \sum_{\Phi} \int d^d x \ \int dt \ \left(\frac{\delta \tilde{\Gamma}^{(n)}}{\delta \Phi} \frac{\delta}{\delta K_{\Phi}} + \frac{\delta \tilde{\Gamma}^{(n)}}{\delta K_{\Phi}} \frac{\delta}{\delta \Phi} \right), \tag{6.46}$$

with

$$\mathcal{D} = \sum_{n=0}^{\infty} g^{2n} \mathcal{D}_n, \tag{6.47}$$

we may rewrite the ZJ equation as

$$\mathcal{D} \tilde{\Gamma} = 0. \tag{6.48}$$

At tree-level, this is simply a statement of the BRST closure of the action:

$$\mathcal{D}_0 \tilde{\Gamma}^{(0)} = 0, \tag{6.49}$$

which forms the base case of our induction. Since Eq. 6.48 holds at all orders in perturbation theory, we expand it and reindex,

$$\mathcal{D}\tilde{\Gamma} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} g^{2m+2n} D_n \tilde{\Gamma}^{(m)} = \sum_{n=0}^{\infty} g^{2n} \sum_{m=0}^{n} D_m \tilde{\Gamma}^{(n-m)} = 0, \tag{6.50}$$

so that for all n we find

$$\sum_{m=0}^{n} \mathcal{D}_m \tilde{\Gamma}^{(n-m)} = 0. \tag{6.51}$$

Now consider the divergent pieces, $\tilde{\Gamma}_{\infty}^{(n)}$, and suppose that our renormalization prescriptions, Eqs. ref, exactly cancel all divergences at n^{th} order. Then at the next order,

$$\mathcal{D}_0\tilde{\Gamma}^{(n+1)} = \mathcal{D}_0\tilde{\Gamma}_{\text{finite}}^{(n+1)} + \mathcal{D}_0\tilde{\Gamma}_{\infty}^{(n+1)} = -\sum_{m=1}^{n+1} \mathcal{D}_m \tilde{\Gamma}^{(n-m+1)}. \tag{6.52}$$

In our inductive hypothesis, however, we assumed that $\tilde{\Gamma}_{\infty}^{(m)} = 0$ for all $m \leq n$. Thus all of the terms on the far right of Eq. 6.52 are finite, and we have

$$\mathcal{D}_0\tilde{\Gamma}_{\infty}^{(n+1)} = 0. \tag{6.53}$$

In principle, $\tilde{\Gamma}_{\infty}^{(n+1)}$ is allowed to have counterterms both in the bulk and on the boundary. The former may be immediately ruled out following the discussion preceding Eq. 6.25. Since external fields in the bulk will always induce at least one Gaussian damping factor, the relevant momentum integrals are convergent. The BPHZ theorem [] then ensures that all boundary subdiagrams are finite as well, as the boundary theory is renormalized. Then there may be no divergent counterterms in the bulk.

This leaves only boundary counterterms, which must be proportional to local products of fields at $t = 0$ with mass dimension $d = 4$ and zero ghost number. The terms containing flow-time derivatives may be discarded, since no product satisfying these restrictions may contain Lagrange multiplier fields, which are required by the action. Since these terms do not appear, Eq. 6.44 in turn excludes the sources K of the variations of such fields. The remaining K_Φ, too, may be ruled out by mass dimension, leaving only the fields A, ψ, $\overline{\psi}$, c, K_A, K_ψ, $K_{\overline{\psi}}$, K_c, and the Lagrange multipliers on the boundaries, $L|_{t=0}$, $\overline{d}|_{t=0}$, $\lambda|_{t=0}$, and $\overline{\lambda}|_{t=0}$. The most general form of this counterterm (see Ref. [], Eqs. 4.48 and 4.59), is determined up to seven formally divergent coefficients: one, w, for the gauge action; one for each the kinetic term, x_1, and the mass term, x_2, in the fermion action; three, y_1, y_2, and y_3, for the sources K_A, $K_{\psi(\overline{\psi})}$, and K_c; and one final constant, z, for the fermionic Lagrange multipliers, which are not ruled out by the ZJ equation. Choosing

$$\begin{aligned}
Z_\psi^{(n+1)} &= x_1 + 2y_2, \\
Z_c^{(n+1)} &= -y_1 - y_3, \\
Z_\chi^{(n+1)} &= x_1 + 4y_2 - 2z, \\
Z_g^{(n+1)} &= w, \\
Z_m^{(n+1)} &= x_2 - x_1, \\
Z_\xi^{(n+1)} &= 2y_1 - w,
\end{aligned} \tag{6.54}$$

exactly cancels all potential divergences in $\tilde{\Gamma}_\infty^{(n+1)}$, and we may conclude that these six renormalization constants are sufficient to negate all divergences to all loop orders.

A couple remarks are now in order concerning the proof outlined above. First, there is a problem in defining sources at vanishing flow time for the fields B, χ, $\overline{\chi}$, and d, since they are constrained on the boundary and are thus not true degrees of freedom. In order to circumvent this ambiguity, the authors of Ref. [] discretized flow time with a forward difference prescription, allowing them to omit the $t = 0$ time slice from the relevant action integrals. This violates the Zinn-Justin equation, but these terms are shown to vanish in the continuum limit.

Second, the bulk gauge fields somehow need no renormalization in the bulk, while the fermions do. This disparity may be clarified by examining the boundary counterterms generated by the flow equations. Consider the (perfectly allowed) counterterm containing the Lagrange multipliers of the gauge sector on the boundary:

$$\tilde{\Gamma}_{L,\overline{d}} = \frac{1}{T_F} \int d^d x \ \operatorname{Tr}\left(z_1 L|_{t=0} A + z_2 \overline{d}|_{t=0} c \right), \tag{6.55}$$

with two divergent coefficients z_1 and z_2. Taking the variations of these fields to the boundary as well, the ZJ equation requires that

$$\begin{aligned}
T_F \mathcal{D}_0 \tilde{\Gamma}_{L,\overline{d}} &= \int d^d x \ \operatorname{Tr}\left[z_1 \delta L|_{t=0} \cdot A + z_1 L|_{t=0} \cdot \delta A + z_2 \delta \overline{d}|_{t=0} \cdot c - z_2 \overline{d}|_{t=0} \cdot \delta c \right] \\
&= \int d^d x \ \operatorname{Tr}\left[(z_1 - z_2) L|_{t=0} \partial c - z_2 \left(L|_{t=0}[A, c] + \overline{d}|_{t=0} c^2 \right) \right]
\end{aligned} \tag{6.56}$$

must vanish (ignoring the fermionic piece of $\delta\overline{d}$ without loss of generality). Since the BRST transformation is inhomogeneous, we must set $z_1 = z_2 = 0$; thus the gauge fields undergo no renormalization in the bulk. On the other hand, for the analogous fermionic counterterm with coefficient z,

$$\tilde{\Gamma}_{\overline{\lambda},\lambda} = z \int d^d x \ \left(\overline{\lambda}|_{t=0} \psi + \overline{\psi} \lambda|_{t=0} \right), \tag{6.57}$$

the variation is itself homogeneous:

$$\begin{aligned}\mathcal{D}_0\tilde{\Gamma}_{\overline{\lambda},\lambda} &= z\int d^dx\ \left(\overline{\lambda}|_{t=0}\psi + \overline{\psi}\lambda|_{t=0}\right)\\ &= z\int d^dx\ \left(\delta\overline{\lambda}|_{t=0}\psi - \overline{\lambda}|_{t=0}\delta\psi + \delta\overline{\psi}\lambda|_{t=0} - \overline{\psi}\delta\lambda|_{t=0}\right)\\ &= z\cdot 0.\end{aligned} \tag{6.58}$$

In this case, since the integral vanishes, there is no condition on z, so the fields λ and $\overline{\lambda}$ indeed produce a counterterm on the boundary (and so require wavefunction renormalization). Recalling that there may be no bulk counterterms like Eq. 6.23, we conclude that the fermions require the inverse renormalization.

Chapter 7

The Short-Flow-Time Expansion

7.1 Composite Operators

In the previous chapter, we found that, due to the Gaussian damping factors induced by the flow equations, the renormalization counterterms of a flowed theory reside exclusively on its boundary. As a result, all correlation functions of the form

$$\begin{aligned}&\langle B^{a_1}_{\alpha_1}(x_1;t_1)\cdots B^{a_n}_{\alpha_n}(x_n;t_n)\overline{\chi}(y_1;s_1)\cdots\overline{\chi}(y_m;s_m)\chi(z_1;u_1)\cdots\chi(z_m;u_m)\rangle\\&\quad = Z_\chi^{-m}\langle B^{a_1}_{\alpha_1}(x_1;t_1)\cdots B^{a_n}_{\alpha_n}(x_n;t_n)\overline{\chi}(y_1;s_1)\cdots\overline{\chi}(y_m;s_m)\chi(z_1;u_1)\cdots\chi(z_m;u_m)\rangle_0\end{aligned} \tag{7.1}$$

are strictly finite at positive flow times provided that the boundary theory is appropriately renormalized. Remarkably, this finiteness carries over to correlation functions as above for which any number of the spacetime coordinates coincide. This follows again from the association of a heat kernel to each flowed field. First observe that the flow does not affect the infrared regime, since all Gaussians tend to unity for small momenta and nonzero flow time, as they must in order to fulfill the boundary conditions. Then we may expect that any IR divergences originate on the boundary. Any other divergences will be ultraviolet, corresponding to the contact of any number of flowed fields at a single point. As the centers of the flowed distributions overlap, the momentum tends to infinity, driving the Gaussians to zero. Functions of the resulting local product of fields will then contain two types of loop integrals. The flowed integrals generated by direct contraction with the operator product are exponentially damped at UV scales by the flowed fields involved, so they converge absolutely. All other loops are radiative corrections to the boundary theory, which are exactly canceled by the boundary counterterms. It follows that for any bare operator

$$\mathcal{O}_0(x;t) = \Gamma B^n(x;t)\overline{\chi}_0^m(x;t)\chi_0^m(x;t), \tag{7.2}$$

where indices are suppressed and all tensor structure is generically represented by Γ, we need only renormalize the fermions (in addition to the boundary parameters, as usual). Then the renormalized operator is simply

$$\mathcal{O}(x;t) = Z_\chi^{-m}\mathcal{O}_0(x;t). \tag{7.3}$$

This allows us to define renormalized correlation functions of local operator products at finite flow time with a simple multiplicative prescription.

7.2 The Short-Flow-Time Expansion

We now have a straightforward and efficient method to renormalize composite operators. If this is to have any predictive power, the flowed matrix elements ought to be relatable to the physical theory at $t = 0$. Of course, as the flow time tends to zero, we expect that the contact divergences of the boundary theory should be recovered, so that all renormalized matrix elements of local operators will in general diverge in this limit. In Ch. **??**, we saw that these divergences could be absorbed into a suitable renormalization of the composite operators by means of the OPE,

$$(\mathcal{O}_i)_0 = Z_{ij}\mathcal{O}_j, \tag{7.4}$$

where the implied sum over j runs over a basis of operators $\mathcal{O}_j$ restricted only by the quantum numbers of $\mathcal{O}_i$. In this case, the equivalence is meant to be interpreted in the limit that the coordinates of all fields in the operator coincide. The infinite constants Z_{ij} contain the contact divergences generated in this limit. Under the flow, the contact terms are smeared with the fields as functions of the flow time t. Following the same arguments, we may write an analogous asymptotic expansion for renormalized flowed operators near the boundary:

$$\mathcal{O}_i(x;t) \overset{t\to 0}{\sim} c_{ij}(t)\mathcal{O}_j(x), \tag{7.5}$$

called the short flow time expansion (SFTE). On the right-hand side, all flow-time-dependence is isolated within the Wilson coefficients $c_{ij}(t)$. By purely dimensional arguments, we may determine their leading-order scaling with the flow time:

$$[c_{ij}(t)] = [\mathcal{O}_i(x;t)] - [\mathcal{O}_j(x)] = d_i - d_j, \tag{7.6}$$

so that

$$c_{ij}(t) \propto t^{\frac{d_j - d_i}{2}}. \tag{7.7}$$

In the event that $d_i = d_j$, the Wilson coefficient diverges logarithmically with t. The more interesting cases, however, are when $d_i > d_j$, and the dependence on t of the mixing coefficients goes as an inverse power of the flow time. These power divergences are typically absent from perturbation theory with dimensional regularization, since they are generated by integrals which become scaleless on the boundary. This is particularly attractive to lattice applications, because the mixing coefficients are decoupled from the lattice regulator at leading order. Of course, in a discretized setting, there may be subleading corrections which depend on the lattice spacing, but these vanish in the continuum limit. In the remaining case, $d_i < d_j$, the Wilson coefficient is suppressed by some positive power of t and vanishes on the boundary. These terms correspond to irrelevant operators and will be hereafter neglected; we will truncate the sum at the logarithmic order and write the irrelevant contributions as an error of $\mathcal{O}(t)$.

7.3 Wilson Coefficients

Since the SFTE is an operator-level relation, we are afforded a considerable amount of freedom in choosing probes for the Wilson coefficients. Specifically, we may choose any external fields with any kinematics to construct matrix elements of the flowed operator. Choosing an operator as in Eq. 7.2 and some external state with generic flowed or unflowed fields Φ_k, we define the renormalized correlation function

$$\begin{aligned}\Gamma_i(x, y_1, ..., y_n; t, s_1, ..., s_n) &\equiv \langle \Phi_1(y_1, s_1) \cdots \Phi_n(y_n, s_n) \mathcal{O}_i(x; t) \rangle \\ &= Z_{\Phi_1}^{-1} \cdots Z_{\Phi_n}^{-1} Z_\chi^{-m} \langle \Phi_1(y_1, s_1) \cdots \Phi_n(y_n, s_n) \mathcal{O}_i(x; t) \rangle_0,\end{aligned} \tag{7.8}$$

where, in case $\Phi_k = B$ for some k, we write $Z_B = 1$ identically. With a suitable choice of external states depending on the field structure of the boundary operators, we may can choose which terms at any order contribute to the expansion of the correlation function. Particularly at next-to-leading order, the external states may often be chosen so that matrix elements of the form above vanish entirely for some j. Inserting this expression into the SFTE, we have

$$\langle \Phi_1(y_1, t_1) \cdots \Phi_n(y_n, t_n) \mathcal{O}_i(x; t) \rangle = \sum_j c_{ij}(t) \langle \Phi_1(y_1, t_1) \cdots \Phi_n(y_n, t_n) \mathcal{O}_j(x) \rangle. \tag{7.9}$$

Introducing the shorthand $\Gamma_i(t) = \Gamma_i(x, y_1, ..., y_n; t, s_1, ..., s_n)$ for $t \geq 0$, we may express the SFTE as a loop expansion. Writing

$$\Gamma_i(t) = \sum_{n=0}^{\infty} g^{2n} \Gamma_i^{(n)}(t), \qquad c_{ij}(t) = \sum_{n=0}^{\infty} g^{2n} c_{ij}^{(n)}, \tag{7.10}$$

the expansion assumes the form

$$\sum_{n=0}^{\infty} g^{2n} \Gamma_i^{(n)}(t) = \sum_j \sum_{n=0}^{\infty} g^{2n} c_{ij}^{(n)}(t) \sum_{m=0}^{\infty} g^{2m} \Gamma_j^{(m)}(0) = \sum_j \sum_{0 \geq m \geq n} g^{2n} c_{ij}^{(n-m)}(t) \Gamma_j^{(m)}(0). \tag{7.11}$$

Equating terms of the same order, we have

$$\Gamma_i^{(n)}(t) = \sum_j \sum_{m=0}^{n} c_{ij}^{(n-m)}(t) \Gamma_j^{(m)}(0). \tag{7.12}$$

On the right side, the boundary correlators may be further expanded in an OPE (**??**),

$$\Gamma_j(0) = Z_{jk}^{-1} [\Gamma_k]_0(0), \tag{7.13}$$

with renormalization constants likewise expanded in the coupling:

$$Z_{jk}^{-1} = \sum_{n=0}^{\infty} g^{2n} [Z_{jk}^{-1}]^{(n)}. \tag{7.14}$$

We may then write the n^{th} term of the SFTE as

$$\Gamma_i^{(n)}(t) = \sum_{j,k} \sum_{0 \leq \ell \leq m \leq n} c_{ij}^{(n-m)}(t) [Z_{jk}^{-1}]^{(m-\ell)} [\Gamma_k]_0^{(\ell)}(0). \tag{7.15}$$

The most useful cases within the scope of this work are the tree-level and one-loop expressions at $n = 0, 1$. In the former case, we have the trivial expression

$$\Gamma_i^{(0)}(t) = c_{ij}^{(0)}(t)[Z_{jk}^{-1}]^{(0)}[\Gamma_k]_0^{(0)}(0), \tag{7.16}$$

where the operator sums over j and k are once again made implicit. Using $Z_{jk}^{(0)} = \delta_{jk}$, and noting that the tree-level structures of the flowed and unflowed matrix elements are identical up to kernels attached to the flowed fields (therefore up to terms of at least order t),

$$\Gamma_i^{(0)}(t) = c_{ij}^{(0)}(t)[\Gamma_j]_0^{(0)}(0) = [\Gamma_i]_0^{(0)}(0) + \mathcal{O}(t), \tag{7.17}$$

we conclude that $c_{ij}^{(0)} = \delta_{ij} + \mathcal{O}(t)$. For $n = 1$, we have after some simplification

$$\Gamma_i^{(1)}(t) = \left\{c_{ij}^{(1)}(t) + [Z_{ij}^{-1}]^{(1)}\right\} \cdot [\Gamma_j]_0^{(0)}(0) + [\Gamma_i]_0^{(1)}(0) + \mathcal{O}(t), \tag{7.18}$$

which gives us an easy recipe for calculating the NLO Wilson coefficients. We will take two approaches in calculating the correlation functions in the next Part of this thesis. Many flowed diagrams will be exactly solvable by the same novel method used to calculate the renormalization constants in previous chapters. On the other hand, when we renormalize the topological charge density and gluon chromoelectric dipole moment operators, many of the integrals or sums will be unsolvable with current methods. We will discuss this more in App. **??**. When they are exactly solvable, we calculate every term above. After renormalizing the boundary parameters, the renormalization of any χ fields will take care of all remaining poles on the flowed side. On the expanded side, the poles from the bare one-loop matrix element are cancelled by the boundary counterterm. For the unsolvable cases, it is easiest to use the method of projectors. To proceed, we first choose a set of external states, defining correlation functions Γ_j as above, and rotate to momentum space. We then define as many differential operators $\mathcal{P}_i$ satisfying

$$\mathcal{P}_i\Gamma_j^{(0)} = \delta_{ij}, \tag{7.19}$$

or, in other words, project out the tree level associated with j^{th} operator. The projectors generally contain derivatives with respect to masses and to any momenta related to derivative couplings and traces over all fermionic, Lorentz, and gauge group indices. After the derivatives are taken, all external scales are taken to zero. In order that these traces do not trivially vanish, we also insert appropriate elements of the spacetime and gauge algebras. To ensure orthogonality, we may diagonalize the operator basis. Finally, we normalize to one by dividing out various numerical constants (polynomials in d, group invariants, *etc.*). As an example, consider the qCMDM operator,

$$\mathcal{O}_{CM} = k_{CM}\overline{\psi}\sigma_{\mu\nu}G_{\mu\nu}\psi, \tag{7.20}$$

where k_{CM} is some unimportant normalization constant, and $\sigma_{\mu\nu} = \frac{i}{2}\gamma_{[\mu,\nu]}$. Choosing an external state of two fermions and a gluon, the amputated tree-level result is just the Feynman rule:

$$[\Gamma_{CM}]_0(p,q,r) = \langle\tilde{\psi}(r)\tilde{A}_\alpha^a(q)\tilde{\overline{\psi}}(p)\mathcal{O}_i\rangle_0 = -2ik_{CM}t^a\sigma_{\alpha\beta}q_\beta. \tag{7.21}$$

We now differentiate with respect to q_γ and multiply by $t^a \sigma_{\gamma\alpha}$ so that the traces do not vanish (t^a is traceless, and $\sigma_{\mu\nu}$ is antisymmetric), which determines the normalization:

$$\mathrm{Tr}\left\{ t^a \sigma_{\gamma\alpha} \frac{\partial}{\partial_{q_\gamma}} \left[-2ik_{CM} t^a \sigma_{\alpha\beta} q_\beta \right] \right\} = 2ik_{CM} d(d-1) n_f C_2(F), \tag{7.22}$$

The projector for the qCMDM is then:

$$\mathcal{P}_{CM}[X] = \frac{1}{2ik_{CM} d(d-1) n_f C_2(F)} \mathrm{Tr}\left\{ t^a \sigma_{\gamma\alpha} \frac{\partial}{\partial_{q_\gamma}} X \right\}. \tag{7.23}$$

We will not often worry about orthogonality. Indeed, there is another operator in Sec. **??** which will not vanish when acted upon by $\mathcal{P}_{CM}$, but the pieces are trivial to disentangle, and the derivative is the critical operation.

When we apply a projector to subleading diagrams, since all external scales are neglected, there may be nothing the regulate the infrared region of some loop integrals. At zero flow time, the loop integrals appearing in $[\Gamma_k]_0^{(\ell)}(0)$ for $\ell > 0$ will in general be of the form

$$I_n(0) = \int_p \frac{1}{(p^2)^n} = \frac{2(4\pi)^{-d/2}}{\Gamma(d/2)} \int_0^\infty dp\, p^{d-2n-1} = \frac{2(4\pi)^{-d/2}}{\Gamma(d/2)} \cdot \left. \frac{p^{d-2n}}{d-2n} \right|_{p=0}^{\infty}. \tag{7.24}$$

In the radial form above, it is easy to see that the disjoint domains of convergence are defined by $d > 2n$ in the IR region ($p \to 0$) and $d < 2n$ in the UV ($p \to \infty$). There is no dimensionful parameter in the integrand, but the integral has a mass dimension of $d - 2n$, so we expect it to vanish. This may be achieved by introducing a factor of one into the formal (undefined) integral and expanding as such:

$$I_n(0) = \int_p \frac{1}{(p^2)^n} \left(\frac{p^2+m^2}{p^2+m^2} \right)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} m^{2\ell} \int_p \frac{(p^2)^{k-\ell-n}}{(p^2+m^2)^k}. \tag{7.25}$$

for some integer k. Performing the integral then the sum, we arrive at

$$I_n(0) = \frac{m^{4-2n}}{(4\pi)^2} \left(\frac{4\pi}{m^2} \right)^{2-d/2} \cdot (-1)^n \frac{d-2k}{d-2n} \frac{\sin(k\pi)}{\pi} \frac{\Gamma(d/2-k)\Gamma(k-d/2)}{\Gamma(d/2)}. \tag{7.26}$$

Since k is an integer, the sine and therefore the integral both vanish.

It is more practically useful to recover this result by analytically continuing the dimension. In this case we split the region of integration by some scale Λ instead of inserting a unit:

$$I_n(0) = \frac{2(4\pi)^{-d_+/2}}{\Gamma(d_+/2)} \int_0^\Lambda dp\, p^{d_+-2n-1} + \frac{2(4\pi)^{-d_-/2}}{\Gamma(d_-/2)} \int_\Lambda^\infty dp\, p^{d_--2n-1}. \tag{7.27}$$

in the first integral, we set $d_+ = 2n + \epsilon_{IR}$ with $\epsilon_{IR} > 0$ to regulate the infrared divergence. For the second integral, we contrariwise define $d_- = 2n - \epsilon_{UV}$ with $\epsilon_{UV} > 0$. The integrals evaluate to

$$I_n(0) = \frac{2(4\pi)^{-d_+/2}}{\Gamma(d_+/2)} \frac{\Lambda^{d_+-2n}}{d_+-2n} - \frac{2(4\pi)^{-d_-/2}}{\Gamma(d_-/2)} \frac{\Lambda^{d_+-2n}}{d_+-2n}, \tag{7.28}$$

which is easily expanded to leading order near $d_\pm = 2n$. For $n = 2$, we have

$$I_2(0) = \frac{1}{(4\pi)^2}\left[\frac{1}{\epsilon_{IR}} + \frac{1}{\epsilon_{UV}}\right] + \mathcal{O}(\epsilon_{IR}, \epsilon_{UV}), \tag{7.29}$$

so that continuing $\epsilon_{IR} \to -\epsilon_{UV}$ forces the integral to vanish. For any other value of n, the total trivially vanishes:

$$I_{n\neq 2}(0) = \frac{1}{(4\pi)^2}\left[-\frac{\Lambda^{4-2n}}{n-2} + \frac{\Lambda^{4-2n}}{n-2}\right] + \mathcal{O}(\epsilon_{IR}, \epsilon_{UV}) = \mathcal{O}(\epsilon_{IR}, \epsilon_{UV}). \tag{7.30}$$

This partition of the integrand is especially imporant for computing flowed integrals after projection. In Eq. 7.18, the correlators at zero flow time vanish by the above arguments. The flowed correlation functions typically contain integrals of the form

$$I_n(u) = \int_p \frac{e^{-p^2 u}}{(p^2)^n} = \frac{2(4\pi)^{-d/2}}{\Gamma(d/2)} \int_0^\infty dp\; e^{-}{-p^2 t p}^{d-2n-1}, \tag{7.31}$$

where u is some nonnegative parameter dependent on the flow time that endows the integral with a scale and damps the UV modes. For $d > 2n$, this is a simple gamma function:

$$I_{n<d/2}(u) = (4\pi t)^{2-d/2} \frac{t^{n-2}}{(4\pi)^2} \frac{\Gamma d/2 - n}{\Gamma(d/2)}. \tag{7.32}$$

For $d \leq 2n$, however, we encounter another IR divergence, so we use. $d = d_+$. Expanding the integral, we have

$$I_{n\geq d/2}(u) = -\frac{1}{(4\pi)^2}\left[\frac{1}{\epsilon_{IR}} - \log(4\pi u) - 1 + \mathcal{O}(\epsilon_{IR})\right], \tag{7.33}$$

evincing the IR pole. These should cancel the the UV pole from any necessary renormalization factors and the boundary counterterm proportional to $[Z_{ij}^{-1}]^{(1)}$, but the signs are wrong. We may then define the boundary integrals as in Eq. 7.29 before analytically continuing, which reintroduces all UV poles and cancel the IR poles. Equivalently, we can simply set $\epsilon_{IR} = -\epsilon_{UV}$ in all calculations, and the poles manifestly cancel.

Now that we have two methods for calculating loop integrals, we can rearrange the SFTE at one-loop order, expressing the Wilson coefficient in terms of quantities we can now calculate:

$$c_{ij}^{(1)}(t)[\Gamma_j]_0^{(0)}(0) = \Gamma_i^{(1)}(t) - [\Gamma_i]_0^{(1)}(0) - [Z_{ij}^{-1}]^{(1)}[\Gamma_j]_0^{(0)}(0) + \mathcal{O}(t). \tag{7.34}$$

The mixing coefficients are then easily readable as the (finite) coefficients of the tree-levels for each operator j.

Part III

CP-Violating Operator Mixing

Chapter 8

Operator Basis

8.1 Hilbert Series

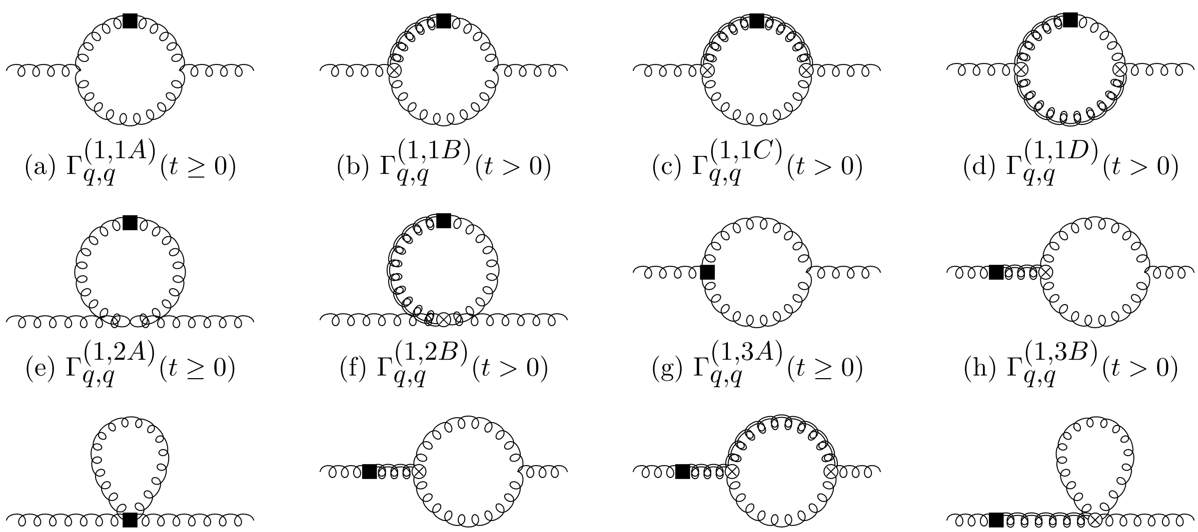
8.2 Mixing Structure

8.3 Chiral Symmetry

Chapter 9

Results

9.1 Topological Charge Density



(a) $\Gamma^{(1,1A)}_{q,q}(t \geq 0)$ (b) $\Gamma^{(1,1B)}_{q,q}(t > 0)$ (c) $\Gamma^{(1,1C)}_{q,q}(t > 0)$ (d) $\Gamma^{(1,1D)}_{q,q}(t > 0)$

(e) $\Gamma^{(1,2A)}_{q,q}(t \geq 0)$ (f) $\Gamma^{(1,2B)}_{q,q}(t > 0)$ (g) $\Gamma^{(1,3A)}_{q,q}(t \geq 0)$ (h) $\Gamma^{(1,3B)}_{q,q}(t > 0)$

(i) $\Gamma^{(1,4)}_{q,q}(t \geq 0)$ (j) $\Gamma^{(1,5A)}_{q,q}(t > 0)$ (k) $\Gamma^{(1,5B)}_{q,q}(t > 0)$ (l) $\Gamma^{(1,5C)}_{q,q}(t > 0)$

Figure 9.1: All topologically distinct contributions to $\Gamma^{(1)}_{q,q}(t \geq 0)$

9.2 Quark Chromoelectric Dipole Moment

We now consider renormalization of the quark chromoelectric- and chromomagnetic-dipole-moment operators (qCEDM and qCMDM) at one-loop order and at positive flow time. These operators are defined as

$$\mathcal{O}_{CE} = k_{CE}\bar{\psi}\tilde{\sigma}_{\mu\nu}G_{\mu\nu}\psi, \tag{9.1a}$$

$$\mathcal{O}_{CM} = k_{CM}\bar{\psi}\sigma_{\mu\nu}G_{\mu\nu}\psi, \tag{9.1b}$$

where k_i are generic normalization constants; F and G are, respectively, the $U(1)$ and $SU(N_C)$ curvature tensors:

$$F_{\mu\nu} = \partial_{[\mu}A_{\nu]}, \tag{9.2a}$$

$$G_{\mu\nu} = t^a G^a_{\mu\nu} = t^a \left\{ \partial_{[\mu} G^a_{\nu]} + f^{abc} G^b_\mu G^c_\nu \right\}; \tag{9.2b}$$

and σ and $\tilde{\sigma}$ are the unit tensor and pseudotensor elements of the d-dimensional STA:

$$\sigma_{\mu\nu} = \frac{i}{2}\gamma_{[\mu,\nu]}, \tag{9.3a}$$

$$\tilde{\sigma}_{\mu\nu} = \frac{1}{2}\left\{\sigma_{\mu\nu}, \gamma_5\right\} \xrightarrow{d\to 4^\pm} \sigma_{\mu\nu}\gamma_5; \tag{9.3b}$$

At finite flow time near the boundary, the renormalized operators are expressible as OPEs in the flow time (SFTEs):

$$\mathcal{O}^R_i(t) = \sum_{\mathcal{O}_j \in \mathcal{A}_i} c_{ij}(t)\mathcal{O}^R_j(0), \tag{9.4}$$

where $\mathcal{A}_i$ is a basis of operators at $t = 0$ with the same quantum numbers as $\mathcal{O}^R_i$. For the dipole-moment operators above, these read

$$\mathcal{O}^R_{CE}(t) = c_{CE,P}(t)\mathcal{O}^R_P(0) + c_{CE,E}(t)\mathcal{O}^R_E(0) + c_{CE,CE}(t)\mathcal{O}^R_{CE}(0) + \mathcal{O}(m,t), \tag{9.5a}$$

$$\mathcal{O}^R_{CM}(t) = c_{CM,S}(t)\mathcal{O}^R_S(0) + c_{CM,M}(t)\mathcal{O}^R_M(0) + c_{CM,CM}(t)\mathcal{O}^R_{CM}(0) + \mathcal{O}(m,t), \tag{9.5b}$$

where the scalar, pseudoscalar, qEDM, and qMDM operators are defined by

$$\mathcal{O}_S = k_S\bar{\psi}\psi, \tag{9.6a}$$

$$\mathcal{O}_P = k_P\bar{\psi}\gamma_5\psi, \tag{9.6b}$$

$$\mathcal{O}_E = k_E\bar{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi, \tag{9.6c}$$

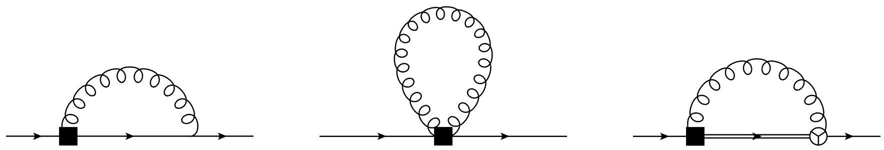
$$\mathcal{O}_M = k_M\bar{\psi}\tilde{\sigma}_{\mu\nu}F_{\mu\nu}\psi. \tag{9.6d}$$

For now, we neglect the quark mass. There are a handful of operators which contribute only at finite mass; these will be studied in a later section.

9.2.1 Mixing With the Pseudoscalar Density: $c_{CE,P}(t)$

Choosing two quark fields as external states, we implicitly define the following correlation functions $\Gamma_i(p;t)$:

$$(2\pi)^d\delta^{(d)}(p+p')\Gamma^R_{i,P}(p;t) = \int_{xyz} e^{-ipx}e^{-ip'y}\langle\psi^R(y)\mathcal{O}^R_i(z;t)\bar{\psi}^R(x)\rangle. \tag{9.7}$$



(a) $\Gamma^{(1,1)}_{CE,P}(t \geq 0)$ (b) $\Gamma^{(1,2)}_{CE,P}(t \geq 0)$ (c) $\Gamma^{(1,3)}_{CE,P}(t > 0)$

Figure 9.2: All topologically distinct contributions to $\Gamma^{(1)}_{CE,P}(t \geq 0)$

At zero electromagnetic coupling, expanding Eq. 9.5a to $\mathcal{O}(g^2)$, we have

$$\begin{aligned}\Gamma^{(0)R}_{CE,P}(t) + g^2\Gamma^{(1)R}_{CE,P}(t) = &\left[c^{(0)}_{CE,P}(t) + g^2 c^{(1)}_{CE,P}(t)\right]\left[\Gamma^{(0)R}_{P,P}(0) + g^2\Gamma^{(1)R}_{P,P}(0)\right] \\ &+ \left[c^{(0)}_{CE,CE}(t) + g^2 c^{(1)}_{CE,CE}(t)\right]\left[\Gamma^{(0)R}_{CE,P}(0) + g^2\Gamma^{(1)R}_{CE,P}(0)\right] + \mathcal{O}(g^4, t) \\ =&\left[c^{(0)}_{CE,P}(t)\Gamma^{(0)R}_{P,P}(0) + c^{(0)}_{CE,CE}(t)\Gamma^{(0)R}_{CE,P}(0)\right] \\ &+g^2\left[c^{(0)}_{CE,P}(t)\Gamma^{(1)R}_{P,P}(0) + c^{(1)}_{CE,P}(t)\Gamma^{(0)R}_{P,P}(0)\right. \\ &\quad\left.+c^{(0)}_{CE,CE}(t)\Gamma^{(1)R}_{CE,P}(0) + c^{(1)}_{CE,CE}(t)\Gamma^{(0)R}_{CE,P}(0)\right] \\ &+ \mathcal{O}(g^4, m, t)s\end{aligned} \tag{9.8}$$

Collecting like powers in the strong coupling and discarding correlation functions that vanish trivially, we have

$$0 = c^{(0)}_{CE,P}(t)\Gamma^{(0)R}_{P,P}(0) + \mathcal{O}(m, t), \tag{9.9a}$$

$$\Gamma^{(1)R}_{CE,P}(t) = c^{(0)}_{CE,P}(t)\Gamma^{(1)R}_{P,P}(0) + c^{(1)}_{CE,P}(t)\Gamma^{(0)R}_{P,P}(0) + c^{(0)}_{CE,CE}(t)\Gamma^{(1)R}_{CE,P}(0) + \mathcal{O}(m, t). \tag{9.9b}$$

Eq. 9.9a enforces

$$c^{(0)}_{CE,P}(t) = 0 + \mathcal{O}(m, t), \tag{9.10}$$

and, choosing external states as in Sec. 9.2.4, we can easily see that

$$c^{(0)}_{CE,CE}(t) = 1 + \mathcal{O}(m, t). \tag{9.11}$$

We are then left with

$$\Gamma^{(1)R}_{CE,P}(t) = c^{(1)}_{CE,P}(t)\Gamma^{(0)R}_{P,P}(0) + \Gamma^{(1)R}_{CE,P}(0) + \mathcal{O}(m, t). \tag{9.12}$$

On the lefthand side, there are three Feynman graphs which contribute to $\Gamma^{(1)}_{CE}(t)$, shown in Fig. 9.2:

$$\Gamma^{(1,1)}_{CE,P}(t) = 3i\frac{k_{CE}}{k_P}\frac{C_2(F)}{(4\pi)^2}\left\{\frac{1}{t} + p^2\left[\log\left(2p^2 t\right) + \gamma_E - \frac{11}{4}\right]\right\}\gamma_5 + \mathcal{O}(m, t), \tag{9.13a}$$

$$\Gamma^{(1,2)}_{CE,P}(t) = 0 + \mathcal{O}(m,t), \tag{9.13b}$$

$$\Gamma^{(1,3)}_{CE,P}(t) = 0 + \mathcal{O}(m,t). \tag{9.13c}$$

Of course, diagrams 9.2a and 9.2c each have a twin diagram under the exchange of the position of the qCEDM vertex with the QCD or flow vertex. The results are identical under the interchange $p \leftrightarrow p'$, so that

$$\Gamma^{(1)}_{CE,P}(t) = 2\Gamma^{(1,1)}_{CE,P}(t) = 6ik_{CE}\frac{C_2(F)}{(4\pi)^2}\left\{\frac{1}{t} + p^2\left[\log\left(2p^2t\right) + \gamma_E - \frac{11}{4}\right]\right\}\gamma_5 + \mathcal{O}(m,t). \tag{9.14}$$

Notice the term proportional to p^2 in brackets. Since we are working at zero mass, we encounter the off-shell operator

$$\mathcal{O}_{\partial^2 P} = k_{\partial^2 P}\bar{\psi}\gamma_5 \overset{\leftrightarrow}{\partial^2} \psi. \tag{9.15}$$

This term leads to the mixing of the qCEDM with the off-shell operator, but it clearly vanishes as we send p^2 to zero. Since the SFTE is insensitive to our kinematics, we may for now choose to put the quarks on shell, so that the subtraction of the pseudoscalar coefficient is cleaner.

For the righthand side of Eq. 9.12, there is a single graph for each term. The pseudoscalar term is tree-level and therefore trivial; it will be modded out of the final result to solve for the Wilson coefficient. The second term receives a contribution from a a pair of diagrams topologically identical to Fig. 9.2a. We find

$$\Gamma^{(0)}_{P,P}(0) = k_P\gamma_5, \tag{9.16a}$$

$$\Gamma^{(1)}_{CE,P}(0) = -6ik_{CE}\frac{C_2(F)}{(4\pi)^2}\left\{\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E}p^2}\right) + \frac{4}{3} + \frac{10}{9}\delta_{HV}\right]\right\}p^2\gamma_5 + \mathcal{O}(m,t). \tag{9.16b}$$

The second term is purely off-shell, so we set it to zero for now, and we may solve for the pseudoscalar mixing coefficient:

$$c^{(1)}_{CE,P}(t) = 6i\frac{k_{CE}}{k_P}\frac{C_2(F)}{(4\pi)^2}\frac{1}{t} + \mathcal{O}(m,t). \tag{9.17}$$

or

$$c_{CE,P}(t) = 6iC_2(F)\frac{k_{CE}}{k_P}\frac{g^2}{(4\pi)^2}\frac{1}{t} + \mathcal{O}(g^4,m,t). \tag{9.18}$$

Returning to the off-shell SFTE and subtracting the pseudoscalar piece from both sides, we are left with only the off-shell pieces of the correlation functions; taking their difference modulo the tree-level gives us the off-shell mixing coefficient:

$$c^{(1)}_{CE,\partial^2 P}(t) = 6i\frac{k_{CE}}{k_{\partial^2 P}}\frac{C_2(F)}{(4\pi)^2}\left\{\log\left(2\bar{\mu}^2 t\right) + \gamma_E - \frac{17}{12} + \frac{10}{9}\delta_{HV}\right\} + \mathcal{O}(m,t). \tag{9.19}$$

9.2.2 Mixing With the Topological Charge Density: $c_{CE,q}(t)$

Choosing now two gluon fields as external states, we recycle our notation $\Gamma_i(q;t)$:

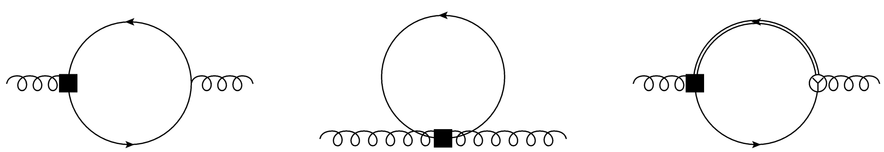
$$(2\pi)^d\delta^{(d)}(q+q')\Gamma_i(q;t) = \int_{xy} e^{-iqx}e^{-iq'y}\langle G^{bR}_\beta(y)\mathcal{O}^R_i(t)G^{aR}_\alpha(x)\rangle. \tag{9.20}$$

Repeating the expansion and reduction as in Sec. ??, we have

$$\Gamma^{(1)}_{CE}(t) = c^{(1)}_{CE,q}(t)\Gamma^{(0)}_q(0) + \Gamma^{(1)}_{CE}(0) + \mathcal{O}(t). \tag{9.21}$$

where the identity, scalar, pseudoscalar, gluon energy density, topological charge density, qEDM, and qMDM operators are defined by

$$\mathcal{O}_I = k_I, \tag{9.22a}$$
$$\mathcal{O}_S = k_S\bar{\psi}\psi, \tag{9.22b}$$
$$\mathcal{O}_P = k_P\bar{\psi}\gamma_5\psi, \tag{9.22c}$$
$$\mathcal{O}_q = k_q \operatorname{Tr}\left\{G_{\mu\nu}\tilde{G}_{\mu\nu}\right\}, \tag{9.22d}$$
$$\mathcal{O}_g = k_g \operatorname{Tr}\left\{G_{\mu\nu}G_{\mu\nu}\right\}, \tag{9.22e}$$
$$\mathcal{O}_E = k_E\bar{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi, \tag{9.22f}$$
$$\mathcal{O}_M = k_M\bar{\psi}\tilde{\sigma}_{\mu\nu}F_{\mu\nu}\psi. \tag{9.22g}$$



(a) $\Gamma^{(1,1)}_{CE,q}(t\geq 0)$ (b) $\Gamma^{(1,2)}_{CE,q}(t\geq 0)$ (c) $\Gamma^{(1,3)}_{CE,q}(t>0)$

Figure 9.3: All distinct contributions to $\Gamma^{(1)}_{CE,q}(t\geq 0)$

9.2.3 Mixing With the Quark Electric Dipole Moment: $c_{CE,E}(t)$

Choosing now two quark fields and a single nondynamical photon as external states, we recycle our notation Γ:

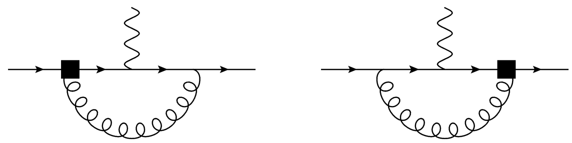
$$(2\pi)^d\delta^{(d)}(p+q-r)\Gamma^R_{i,E}(p,r;t) = \int_{wxyz} e^{-ipw}e^{-iqx}e^{iry}\langle\psi^R(y)\mathcal{O}^R_i(z;t)A^R_\alpha(x)\bar{\psi}^R(w)\rangle. \tag{9.23}$$

Repeating the expansion and reduction as in Sec. 9.2.1, we have

$$\Gamma^{(1)R}_{CE,E}(t) = c^{(1)}_{CE,P}(t)\Gamma^{(0)R}_{P,E}(0) + c^{(1)}_{CE,E}(t)\Gamma^{(0)R}_{E,E}(0) + \Gamma^{(1)R}_{CE,E}(0) + \mathcal{O}(t). \tag{9.24}$$

This time there are extra diagrams on the left that exactly cancel the pseudoscalar term on the right. These are non-1PI, so we may study an equivalent equation where the correlators are strictly 1PI;

$$\Gamma^{(1)R}_{CE,E}(t) = c^{(1)}_{CE,E}(t)\Gamma^{(0)R}_{E,E}(0) + \Gamma^{(1)R}_{CE,E}(0) + \mathcal{O}(t). \tag{9.25}$$



(a) $\Gamma^{(1,1a)}_{CE,E}(t \geq 0)$ (b) $\Gamma^{(1,1b)}_{CE,E}(t \geq 0)$

Figure 9.4: All 1PI contributions to $\Gamma^{(1)}_{CE,E}(t \geq 0)$

There is one pair of diagrams for each term, shown in Fig. 9.4. Since the mixing is linear in the photon momentum q_α, we are free to discard other soft scales; the IR divergences will be regulated by this momentum so long as the outgoing quark is kept off-shell. This, however, breaks the symmetry under the exchange of quark indices, and we must evaluate the graphs in both Figs. 9.4a and 9.4b independently. Further, this choice of kinematics introduces another "nuisance" operator:

$$\mathcal{O}_N = k_N \bar{\psi}_E \gamma_5 \psi_E, \tag{9.26}$$

where the equation-of-motion fields

$$\psi_E = (\not{D} + m)\psi, \tag{9.27a}$$

$$\bar{\psi}_E = \bar{\psi}(\overleftarrow{\not{D}} - m) \tag{9.27b}$$

vanish on the mass shell. At $(p, r) = (0, q)$, there are two tensors which appear within this calculation, $t^a q_\alpha \gamma_5$ and $t^a \sigma_{\alpha\beta}\gamma_5 q_\beta$, which are related to the tree-level correlators by

$$t^a q_\alpha \gamma_5 = \frac{1}{2k_E}\Gamma^{(0)}_E + \frac{i}{k_N}\Gamma^{(0)}_N, \tag{9.28a}$$

$$t^a \sigma_{\alpha\beta}\gamma_5 q_\beta = \frac{i}{2k_E}\Gamma^{(0)}_E. \tag{9.28b}$$

At positive flow time,

$$\begin{aligned}\Gamma^{(1,1a)}_{CE,E}(t) = 2\frac{C_2(F)}{(4\pi)^2}\Bigg\{ &\left[\log\left(2q^2 t\right) + \gamma_E - 1\right]\Gamma^{(0)}_E \\ &+ \frac{3}{2}i\frac{k_{CE}}{k_N}\left[\log\left(2q^2 t\right) + \gamma_E - \frac{3}{2}\right]\Gamma^{(0)}_N\Bigg\} + \mathcal{O}(m, \epsilon),\end{aligned} \tag{9.29a}$$

$$\begin{aligned}\Gamma^{(1,1b)}_{CE,E}(t) = 2\frac{C_2(F)}{(4\pi)^2}\Bigg\{ &\left[\log\left(2q^2 t\right) + \gamma_E - 1\right]\Gamma^{(0)}_E \\ &+ \frac{3}{2}i\frac{k_{CE}}{k_N}\left[\log\left(2q^2 t\right) + \gamma_E - 1\right]\Gamma^{(0)}_N\Bigg\} + \mathcal{O}(m, \epsilon).\end{aligned} \tag{9.29b}$$

On the boundary,

$$\Gamma^{(1,1a)}_{CE,E}(0) = -2\frac{C_2(F)}{(4\pi)^2}\Bigg\{ \left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E}p^2}\right) + \frac{3}{2} + \frac{1}{3}\delta_{HV}\right]\Gamma^{(0)}_E + \frac{3}{2}i\frac{k_{CE}}{k_N}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E}q^2}\right) + \frac{4}{3} + \frac{10}{9}\delta_{HV}\right]\Gamma^{(0)}_N\Bigg\} + \mathcal{O}(m,\epsilon), \tag{9.30a}$$

$$\Gamma^{(1,1b)}_{CE,E}(0) = -2\frac{C_2(F)}{(4\pi)^2}\Bigg\{ \left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E}p^2}\right) + 2 + \frac{1}{3}\delta_{HV}\right]\Gamma^{(0)}_E + \frac{3}{2}i\frac{k_{CE}}{k_N}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E}q^2}\right) + \frac{4}{3} + \frac{10}{9}\delta_{HV}\right]\Gamma^{(0)}_N\Bigg\} + \mathcal{O}(m,\epsilon), \tag{9.30b}$$

and the Wilson coefficients are

$$c_{CE,E}(t) = 4C_2(F)\frac{g^2}{(4\pi)^2}\left\{\log\left(2\bar{\mu}^2 t\right) + \gamma_E + \frac{3}{2} + \frac{2}{3}\delta_{HV}\right\} + \mathcal{O}(g^4, m, t), \tag{9.31a}$$

$$c_{CE,N}(t) = 4C_2(F)\frac{g^2}{(4\pi)^2}\left\{\log\left(2\bar{\mu}^2 t\right) + \gamma_E + \frac{1}{6} + \frac{20}{9}\delta_{HV}\right\} + \mathcal{O}(g^4, m, t). \tag{9.31b}$$

9.2.4 Mixing With the Quark Chromoelectric Dipole Moment: $c_{CE,CE}(t)$

We now choose two quark fields and a single gluon as external states and again redefine Γ:

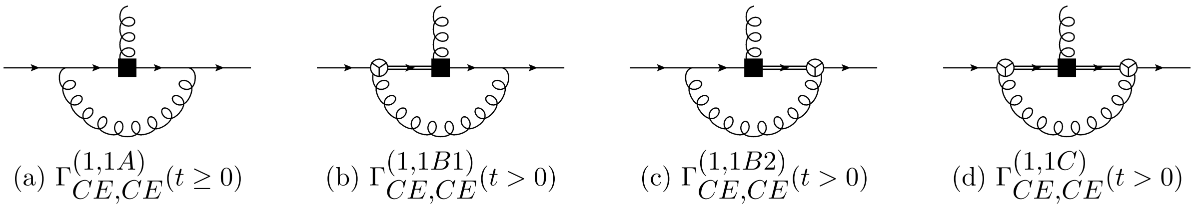
$$(2\pi)^d\delta^{(d)}(p+q-r)\Gamma^R_{i,CE}(p,r;t) = \int_{wxyz} e^{-ipw}e^{-iqx}e^{iry}\langle\psi^R(y)\mathcal{O}^R_i(z;t)G^{aR}_\alpha(x)\bar{\psi}^R(w)\rangle. \tag{9.32}$$

Reducing the SFTE, we may subtract all one-particle reducible diagrams from both sides, so that all pseudoscalar and qEDM terms cancel, leaving us with

$$\Gamma^{(1)R}_{CE,CE}(t) = c^{(1)}_{CE,CE}(t)\Gamma^{(0)R}_{CE,CE}(0) + \Gamma^{(1)R}_{CE,CE}(0) + \mathcal{O}(t). \tag{9.33}$$

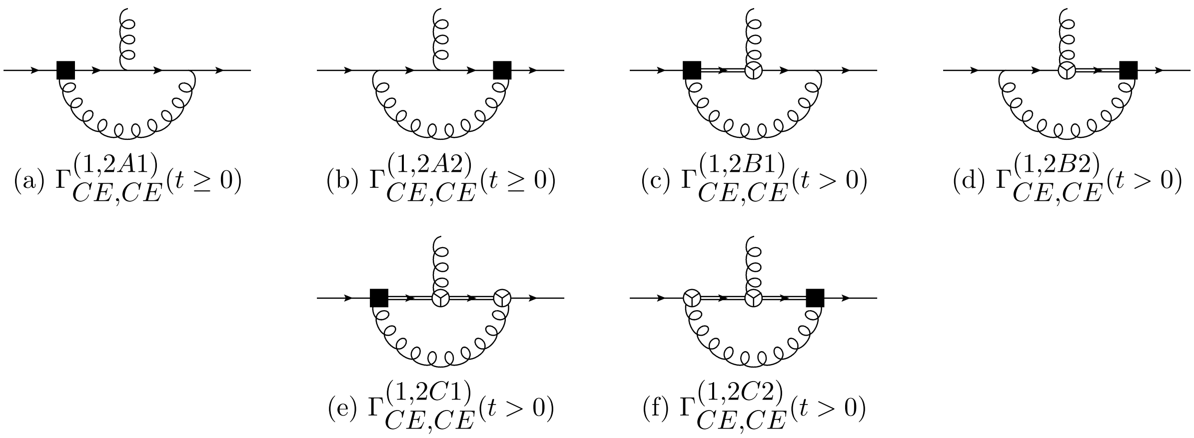
The one-loop flowed correlator produces thirty-four 1PI diagrams, which are shown in Figs. 9.5-9.11, as well as a handful of unique topologies related to the renormalization of the strong coupling and the fermion fields at positive flow time, shown in Figs. **??** and **??**. These latter diagrams have poles at $d = 4$ which are renormalized away; however, they also contain logarithms and finite pieces which ultimately contribute to the self-mixing coefficient for the qCEDM. The 1PI diagrams are collected into classes by the structure of their radiative corrections. Classes $1-5$ consist of the topologies that exist on the boundary, along with their corrections derived from higher-order terms in the flow equations. Classes 6 and 7 are 1PI diagrams that exist only in the bulk.

As with the qEDM mixing, if we set $(p,r) = (0,q)$, we have a simpler calculation with no loss of information on the mixing of on- or off-shell operators at the expense of having to calculate all diagrams, including mirror-images. The diagrams are labeled $\Gamma^{(1,XYZ)}_{CE,CE}$: X is the



(a) $\Gamma^{(1,1A)}_{CE,CE}(t \geq 0)$ (b) $\Gamma^{(1,1B1)}_{CE,CE}(t > 0)$ (c) $\Gamma^{(1,1B2)}_{CE,CE}(t > 0)$ (d) $\Gamma^{(1,1C)}_{CE,CE}(t > 0)$

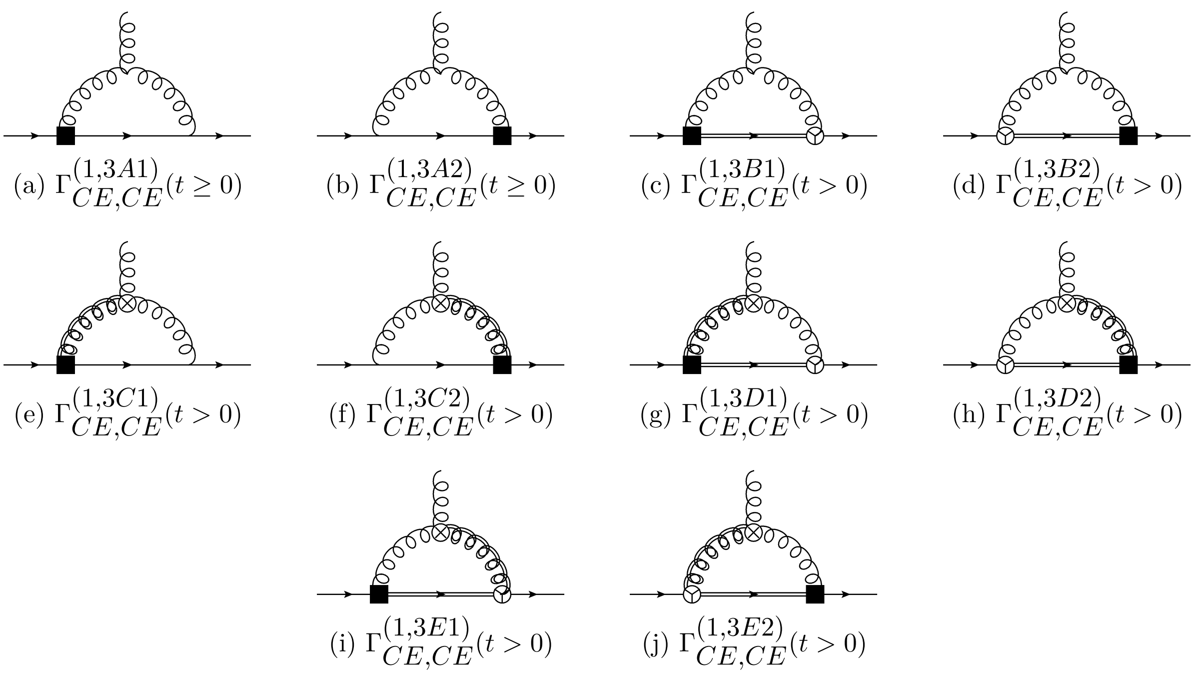
Figure 9.5: All contributions to $\Gamma^{(1,1)}_{CE,CE}(t \geq 0)$



(a) $\Gamma^{(1,2A1)}_{CE,CE}(t \geq 0)$ (b) $\Gamma^{(1,2A2)}_{CE,CE}(t \geq 0)$ (c) $\Gamma^{(1,2B1)}_{CE,CE}(t > 0)$ (d) $\Gamma^{(1,2B2)}_{CE,CE}(t > 0)$

(e) $\Gamma^{(1,2C1)}_{CE,CE}(t > 0)$ (f) $\Gamma^{(1,2C2)}_{CE,CE}(t > 0)$

Figure 9.6: All contributions to $\Gamma^{(1,2)}_{CE,CE}(t \geq 0)$



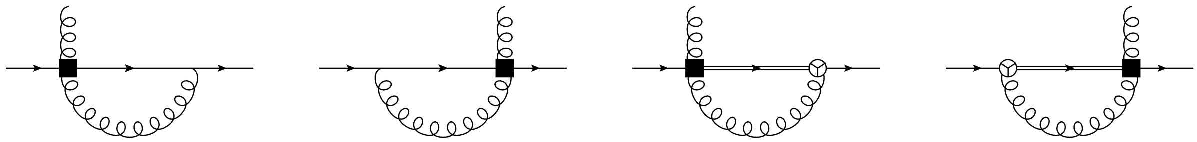
(a) $\Gamma^{(1,3A1)}_{CE,CE}(t \geq 0)$ (b) $\Gamma^{(1,3A2)}_{CE,CE}(t \geq 0)$ (c) $\Gamma^{(1,3B1)}_{CE,CE}(t > 0)$ (d) $\Gamma^{(1,3B2)}_{CE,CE}(t > 0)$

(e) $\Gamma^{(1,3C1)}_{CE,CE}(t > 0)$ (f) $\Gamma^{(1,3C2)}_{CE,CE}(t > 0)$ (g) $\Gamma^{(1,3D1)}_{CE,CE}(t > 0)$ (h) $\Gamma^{(1,3D2)}_{CE,CE}(t > 0)$

(i) $\Gamma^{(1,3E1)}_{CE,CE}(t > 0)$ (j) $\Gamma^{(1,3E2)}_{CE,CE}(t > 0)$

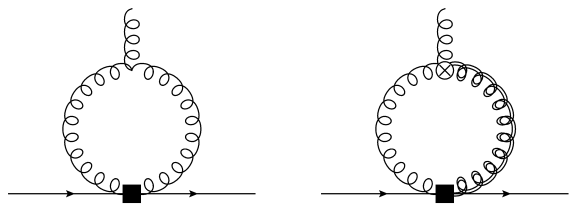
Figure 9.7: All contributions to $\Gamma^{(1,3)}_{CE,CE}(t \geq 0)$

class; Y is the diagram within that class; and Z is the orientation, with 1 having the operator directly connected to the $\bar{\psi}$ field and 2 having the operator insertion on the ψ field. On the



(a) $\Gamma^{(1,4A1)}_{CE,CE}(t \geq 0)$ (b) $\Gamma^{(1,4A2)}_{CE,CE}(t \geq 0)$ (c) $\Gamma^{(1,4B1)}_{CE,CE}(t > 0)$ (d) $\Gamma^{(1,4B2)}_{CE,CE}(t > 0)$

Figure 9.8: All contributions to $\Gamma^{(1,4)}_{CE,CE}(t \geq 0)$

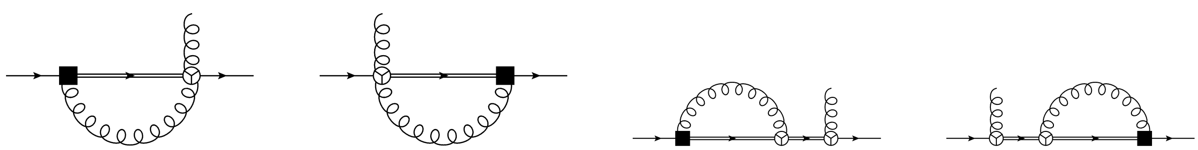


(a) $\Gamma^{(1,5A)}_{CE,CE}(t \geq 0)$ (b) $\Gamma^{(1,5B)}_{CE,CE}(t > 0)$

Figure 9.9: All contributions to $\Gamma^{(1,5)}_{CE,CE}(t \geq 0)$

(a) $\Gamma^{(1,6A1)}_{CE,CE}(t > 0)$ (b) $\Gamma^{(1,6A2)}_{CE,CE}(t > 0)$ (c) $\Gamma^{(1,6B1)}_{CE,CE}(t > 0)$ (d) $\Gamma^{(1,6B2)}_{CE,CE}(t > 0)$

Figure 9.10: All contributions to $\Gamma^{(1,6)}_{CE,CE}(t \geq 0)$



(a) $\Gamma^{(1,7A1)}_{CE,CE}(t > 0)$ (b) $\Gamma^{(1,7A2)}_{CE,CE}(t > 0)$ (c) $\Gamma^{(1,7B1)}_{CE,CE}(t > 0)$ (d) $\Gamma^{(1,7B2)}_{CE,CE}(t > 0)$

Figure 9.11: All contributions to $\Gamma^{(1,7)}_{CE,CE}(t \geq 0)$

flowed side of Eq. 9.33, the momentum of the incoming gluon q regulates all IR divergences, while the flow time regulates the UV for all 1PI diagrams, and no regulator is needed; *viz.*, all diagrams are evaluated directly at $d = 4$. The reducible diagrams (Figs.**??**-**??**) are equal to their counterparts from Secs. **??** and **??** (modulo their tree-levels) times the tree-level qCEDM. Thus, the definition of σ is purely four-dimensional within $\Gamma^{(1)}_{CE,CE}(t > 0)$. The $t = 0$ side loses its UV regulator, and must be evaluated at $d = 4 - 2\epsilon$, where $1 > |\epsilon| > 0$. We must, then, make a choice of prescription for γ_5, or more specifically, $\tilde{\sigma}_{\mu\nu}$. We present results in two γ_5 schemes with three definitions of $\tilde{\sigma}_{\mu\nu}$: naïve dimensional regularization (NDR) and the t'Hooft-Veltman-Breitenlohner-Maison (HVBM) scheme, with the definitions

$$\tilde{\sigma}_{\mu\nu} = \begin{cases} \frac{1}{2}\left\{\sigma_{\mu\nu}, \gamma_5\right\}, & \text{scheme 1;} \quad (9.34a) \\ -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\sigma_{\rho\sigma}, & \text{scheme 2;} \quad (9.34b) \\ -\frac{1}{(d-2)(d-3)}\epsilon_{\mu\nu\rho\sigma}\sigma_{\rho\sigma}, & \text{scheme 3;} \quad (9.34c) \end{cases}$$

all of which coincide in NDR, for which

$$\tilde{\sigma}_{\mu\nu} = \sigma_{\mu\nu}\gamma_5. \tag{9.35}$$

Of course, all conventions yield the same logarithms, but the finite parts are scheme-dependent. To that end, we define

$$\delta^i_{HV} = \begin{cases} 1, & \text{scheme } i, \\ 0, & \text{else.} \end{cases} \tag{9.36}$$

For $t > 0$, we find

$$\Gamma^{(1,1A)}_{CE,CE}(t) = \mathcal{O}(m), \tag{9.37a}$$

$$\Gamma^{(1,1B1)}_{CE,CE}(t) = \mathcal{O}(m), \tag{9.37b}$$

$$\Gamma^{(1,1B2)}_{CE,CE}(t) = \mathcal{O}(m), \tag{9.37c}$$

$$\Gamma^{(1,1C)}_{CE,CE}(t) = \mathcal{O}(m), \tag{9.37d}$$

$$\begin{aligned}\Gamma^{(1,2A1)}_{CE,CE}(t) = -\frac{C_2(A) - 2C_2(F)}{(4\pi)^2} \cdot \Bigg\{ & \left[\log\left(2q^2t\right) + \gamma_E - 1\right]\Gamma^{(0)}_{CE} \\ & + \frac{3}{2}i\frac{k_{CE}}{k_N}\left[\log\left(2q^2t\right) + \gamma_E - \frac{3}{2}\right]\Gamma^{(0)}_N\Bigg\} + \mathcal{O}(m,t),\end{aligned} \tag{9.37e}$$

$$\begin{aligned}\Gamma^{(1,2A2)}_{CE,CE}(t) = -\frac{C_2(A) - 2C_2(F)}{(4\pi)^2} \cdot \Bigg\{ & \left[\log\left(2q^2t\right) + \gamma_E - 1\right]\Gamma^{(0)}_{CE} \\ & + \frac{3}{2}i\frac{k_{CE}}{k_N}\left[\log\left(2q^2t\right) + \gamma_E - 1\right]\Gamma^{(0)}_N\Bigg\} + \mathcal{O}(m,t),\end{aligned} \tag{9.37f}$$

$$\Gamma^{(1,2B1)}_{CE,CE}(t) = \frac{C_2(A) - 2C_2(F)}{(4s\pi)^2} \cdot \left\{\frac{13}{16}\Gamma^{(0)}_{CE} + \frac{15}{8}i\frac{k_{CE}}{k_N}\Gamma^{(0)}_N\right\} + \mathcal{O}(m,t), \tag{9.37g}$$

$$\Gamma^{(1,2B2)}_{CE,CE}(t) = \frac{C_2(A) - 2C_2(F)}{(4\pi)^2} \cdot \left\{-\frac{13}{16}\Gamma^{(0)}_{CE} - \frac{3}{8}i\frac{k_{CE}}{k_N}\Gamma^{(0)}_N\right\} + \mathcal{O}(m,t), \tag{9.37h}$$

$$\Gamma^{(1,2C1)}_{CE,CE}(t) = \frac{C_2(A) - 2C_2(F)}{(4\pi)^2} \cdot \frac{1}{16}\Gamma^{(0)}_{CE} + \mathcal{O}(m,t), \tag{9.37i}$$

$$\Gamma^{(1,2C2)}_{CE,CE}(t) = \mathcal{O}(m), \tag{9.37j}$$

$$\Gamma^{(1,3A1)}_{CE,CE}(t) = -\frac{C_2(A)}{(4\pi)^2} \cdot \left\{ \frac{7}{4} \left[\log\left(2q^2 t\right) + \gamma_E - \frac{3}{14} \right] \Gamma^{(0)}_{CE} + \frac{3}{4} i \frac{k_{CE}}{k_N} \left[\log\left(2q^2 t\right) + \gamma_E + \frac{3}{2} \right] \Gamma^{(0)}_{N} \right\} + \mathcal{O}(m,t), \tag{9.37k}$$

$$\Gamma^{(1,3A2)}_{CE,CE}(t) = -\frac{C_2(A)}{(4\pi)^2} \cdot \left\{ \frac{5}{4} \left[\log\left(2q^2 t\right) + \gamma_E - \frac{1}{5} \right] \Gamma^{(0)}_{CE} - \frac{3}{2} i \frac{k_{CE}}{k_N} \left[\log\left(2q^2 t\right) + \gamma_E - \frac{1}{2} \right] \Gamma^{(0)}_{N} \right\} + \mathcal{O}(m,t), \tag{9.37l}$$

$$\Gamma^{(1,3B1)}_{CE,CE}(t) = \frac{C_2(A)}{(4\pi)^2} \cdot \frac{3}{8} \Gamma^{(0)}_{CE} + \mathcal{O}(m,t), \tag{9.37m}$$

$$\Gamma^{(1,3B2)}_{CE,CE}(t) = \mathcal{O}(m), \tag{9.37n}$$

$$\Gamma^{(1,3C1)}_{CE,CE}(t) = 2\frac{C_2(A)}{(4\pi)^2} \cdot \left\{ \frac{3}{32} \Gamma^{(0)}_{CE} - \frac{9}{16} i \frac{k_{CE}}{k_N} \Gamma^{(0)}_{N} \right\} + \mathcal{O}(m,t), \tag{9.37o}$$

$$\Gamma^{(1,3C2)}_{CE,CE}(t) = 2\frac{C_2(A)}{(4\pi)^2} \cdot \left\{ \frac{11}{32} \Gamma^{(0)}_{CE} - \frac{3}{16} i \frac{k_{CE}}{k_N} \Gamma^{(0)}_{N} \right\} + \mathcal{O}(m,t), \tag{9.37p}$$

$$\Gamma^{(1,3D1)}_{CE,CE}(t) = 2\frac{C_2(A)}{(4\pi)^2} \cdot \frac{1}{16} \Gamma^{(0)}_{CE} + \mathcal{O}(m,t), \tag{9.37q}$$

$$\Gamma^{(1,3D2)}_{CE,CE}(t) = \mathcal{O}(m), \tag{9.37r}$$

$$\Gamma^{(1,3E1)}_{CE,CE}(t) = 2\frac{C_2(A)}{(4\pi)^2} \cdot \frac{3}{64} \Gamma^{(0)}_{CE} + \mathcal{O}(m,t), \tag{9.37s}$$

$$\Gamma^{(1,3E2)}_{CE,CE}(t) = \mathcal{O}(m), \tag{9.37t}$$

$$\Gamma^{(1,4A1)}_{CE,CE}(t) = \frac{C_2(A)}{(4\pi)^2} \cdot \left\{ \frac{1}{2} \left[\log\left(2q^2 t\right) + \gamma_E - 1 \right] \Gamma^{(0)}_{CE} + \frac{3}{2} i \frac{k_{CE}}{k_N} \left[\log\left(2q^2 t\right) + \gamma_E - 1 \right] \Gamma^{(0)}_{N} \right\} + \mathcal{O}(m,t), \tag{9.37u}$$

$$\Gamma^{(1,4A2)}_{CE,CE}(t) = \mathcal{O}(m), \tag{9.37v}$$

$$\Gamma^{(1,4B1)}_{CE,CE}(t) = -\frac{C_2(A)}{(4\pi)^2} \cdot \frac{1}{2} \Gamma^{(0)}_{CE} + \mathcal{O}(m,t), \tag{9.37w}$$

$$\Gamma^{(1,4B2)}_{CE,CE}(t) = \mathcal{O}(m), \tag{9.37x}$$

$$\Gamma^{(1,5A)}_{CE,CE}(t) = \frac{1}{2} \frac{C_2(A)}{(4\pi)^2} \cdot 3 \left[\log\left(2q^2 t\right) + \gamma_E - 1 \right] \Gamma^{(0)}_{CE} + \mathcal{O}(m,t), \tag{9.37y}$$

$$\Gamma^{(1,5B)}_{CE,CE}(t) = -2\frac{C_2(A)}{(4\pi)^2} \cdot \frac{25}{32} \Gamma^{(0)}_{CE} + \mathcal{O}(m,t), \tag{9.37z}$$

$$\Gamma^{(1,6A1)}_{CE,CE}(t) = \mathcal{O}(m), \tag{9.37aa}$$

$$\Gamma^{(1,6A2)}_{CE,CE}(t) = \mathcal{O}(m), \tag{9.37ab}$$

$$\Gamma^{(1,6B1)}_{CE,CE}(t) = -\frac{C_2(F)}{(4\pi)^2} \cdot \left\{ 3\Gamma^{(0)}_{CE} + 6i\frac{k_{CE}}{k_N}\Gamma^{(0)}_{N} \right\} + \mathcal{O}(m,t), \tag{9.37ac}$$

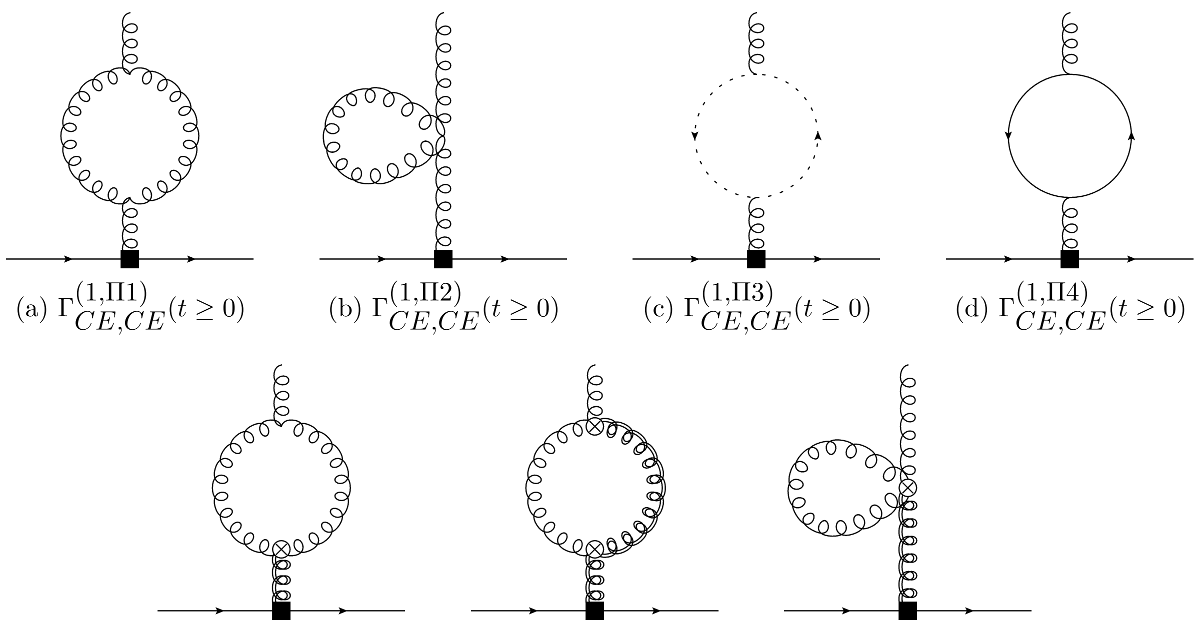
$$\Gamma^{(1,6B2)}_{CE,CE}(t) = \mathcal{O}(m), \tag{9.37ad}$$

$$\Gamma^{(1,7A1)}_{CE,CE}(t) = \mathcal{O}(m), \tag{9.37ae}$$

$$\Gamma^{(1,7A2)}_{CE,CE}(t) = 2\frac{C_2(A) - 4C_2(F)}{(4\pi)^2} \cdot \frac{1}{64}\Gamma^{(0)}_{CE} + \mathcal{O}(m,t), \tag{9.37af}$$

$$\Gamma^{(1,7B1)}_{CE,CE}(t) = \mathcal{O}(m), \tag{9.37ag}$$

$$\Gamma^{(1,7B2)}_{CE,CE}(t) = \mathcal{O}(m). \tag{9.37ah}$$



(a) $\Gamma^{(1,\Pi 1)}_{CE,CE}(t \geq 0)$ (b) $\Gamma^{(1,\Pi 2)}_{CE,CE}(t \geq 0)$ (c) $\Gamma^{(1,\Pi 3)}_{CE,CE}(t \geq 0)$ (d) $\Gamma^{(1,\Pi 4)}_{CE,CE}(t \geq 0)$

(e) $\Gamma^{(1,\Pi 5)}_{CE,CE}(t > 0)$ (f) $\Gamma^{(1,\Pi 6)}_{CE,CE}(t > 0)$ (g) $\Gamma^{(1,\Pi 7)}_{CE,CE}(t > 0)$

The diagrams related to the coupling renormalization and quark field renormalization are easily evaluated:

$$\Gamma^{(1,\Pi 1)}_{CE,CE}(t \geq 0) = \frac{19}{12}\frac{C_2(A)}{(4\pi)^2} \cdot \left\{ \frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2} \right) + \frac{116}{57} \right\}\Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon,t), \tag{9.38a}$$

$$\Gamma^{(1,\Pi 2)}_{CE,CE}(t \geq 0) = \mathcal{O}(\epsilon), \tag{9.38b}$$

$$\Gamma^{(1,\Pi 3)}_{CE,CE}(t \geq 0) = \frac{1}{12}\frac{C_2(A)}{(4\pi)^2} \cdot \left\{ \frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2} \right) + \frac{8}{3} \right\}\Gamma^{(0)}_{CE} + \mathcal{O}\epsilon,(t), \tag{9.38c}$$

$$\Gamma^{(1,\Pi 4)}_{CE,CE}(t \geq 0) = \frac{4}{3}\frac{T_F n_f}{(4\pi)^2} \cdot \left\{ \frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2} \right) + \frac{5}{3} \right\}\Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon,t), \tag{9.38d}$$

$$\Gamma^{(1,\Pi 5)}_{CE,CE}(t > 0) = \frac{3}{2}\frac{C_2(A)}{(4\pi)^2} \cdot \left\{ \frac{1}{\epsilon} + \log(8\pi t) + \frac{5}{6} \right\}\Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon,t), \tag{9.38e}$$

$$\Gamma^{(1,\Pi 6)}_{CE,CE}(t>0) = \frac{C_2(A)}{(4\pi)^2}\cdot\left\{\frac{1}{\epsilon} + \log\left(8\pi t\right) - \frac{1}{4}\right\}\Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon, t), \tag{9.38f}$$

$$\Gamma^{(1,\Pi 7)}_{CE,CE}(t>0) = -\frac{3}{2}\frac{C_2(A)}{(4\pi)^2}\cdot\left\{\frac{1}{\epsilon} + \log\left(8\pi t\right) + \frac{1}{3}\right\}\Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon, t), \tag{9.38g}$$

$$\Gamma^{(1,\Sigma 1a)}_{CE,CE}(t\geq 0) = -\frac{C_2(F)}{(4\pi)^2}\cdot\left\{\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E}q^2}\right) - 1\right\}\Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon, m, t), \tag{9.38h}$$

$$\Gamma^{(1,\Sigma 1b)}_{CE,CE}(t\geq 0) = -\frac{C_2(F)}{(4\pi)^2}\cdot\left\{\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E}p^2}\right) - 1\right\}\Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon, m, t), \tag{9.38i}$$

$$\Gamma^{(1,\Sigma 2a)}_{CE,CE}(t>0) = \frac{C_2(F)}{(4\pi)^2}\cdot\left\{\frac{1}{\epsilon} + \log\left(8\pi t\right) + \gamma_E + 1\right\}\Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon, m, t), \tag{9.38j}$$

$$\Gamma^{(1,\Sigma 2b)}_{CE,CE}(t>0) = \frac{C_2(F)}{(4\pi)^2}\cdot\left\{\frac{1}{\epsilon} + \log\left(8\pi t\right) + \gamma_E + 1\right\}\Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon, m, t), \tag{9.38k}$$

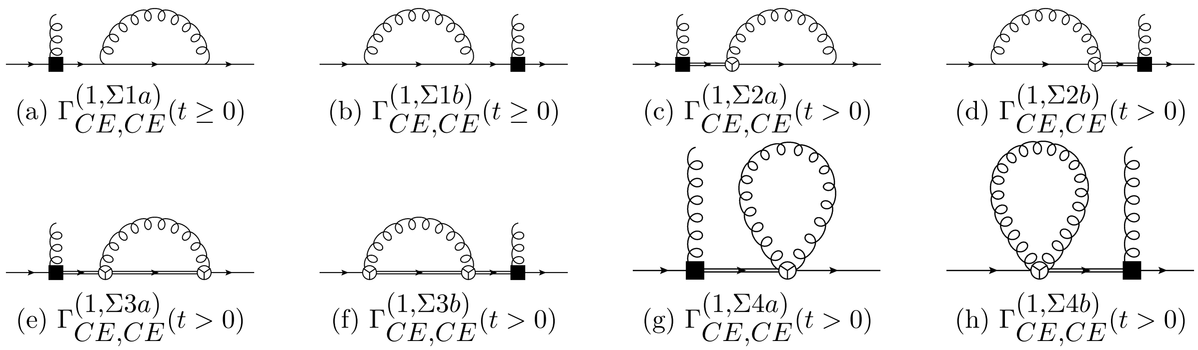
$$\Gamma^{(1,\Sigma 3a)}_{CE,CE}(t>0) = \mathcal{O}(m, t), \tag{9.38l}$$

$$\Gamma^{(1,\Sigma 3b)}_{CE,CE}(t>0) = \mathcal{O}(m, t), \tag{9.38m}$$

$$\Gamma^{(1,\Sigma 4a)}_{CE,CE}(t>0) = -2\frac{C_2(F)}{(4\pi)^2}\cdot\left\{\frac{1}{\epsilon} + \log\left(8\pi t\right) + \gamma_E + \frac{1}{2}\right\}\Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon, m, t), \tag{9.38n}$$

$$\Gamma^{(1,\Sigma 4b)}_{CE,CE}(t>0) = -2\frac{C_2(F)}{(4\pi)^2}\cdot\left\{\frac{1}{\epsilon} + \log\left(8\pi t\right) + \gamma_E + \frac{1}{2}\right\}\Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon, m, t). \tag{9.38o}$$

The diagram in Fig. 9.38i vanishes for $p = 0$, since the loop becomes scaleless. We leave it here, however, so that the pole will be explicitly renormalized by Z_ψ on both sides of the flow equation, taking the $p \to 0$ limit of the Wilson coefficient. (The p-dependence cancels precisely between the two sides.) At $t = 0$, we have



(a) $\Gamma^{(1,\Sigma 1a)}_{CE,CE}(t\geq 0)$ (b) $\Gamma^{(1,\Sigma 1b)}_{CE,CE}(t\geq 0)$ (c) $\Gamma^{(1,\Sigma 2a)}_{CE,CE}(t>0)$ (d) $\Gamma^{(1,\Sigma 2b)}_{CE,CE}(t>0)$

(e) $\Gamma^{(1,\Sigma 3a)}_{CE,CE}(t>0)$ (f) $\Gamma^{(1,\Sigma 3b)}_{CE,CE}(t>0)$ (g) $\Gamma^{(1,\Sigma 4a)}_{CE,CE}(t>0)$ (h) $\Gamma^{(1,\Sigma 4b)}_{CE,CE}(t>0)$

$$\Gamma^{(1,1A)}_{CE,CE}(0) = \mathcal{O}(m), \tag{9.39a}$$

$$\begin{aligned}\Gamma^{(1,2A1)}_{CE,CE}(0) = \frac{C_2(A)-2C_2(F)}{(4\pi)^2}\Bigg\{ &\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E}q^2}\right) + \frac{3}{2} + \frac{1}{3}\delta^1_{HV} + \frac{1}{2}\delta^2_{HV} + \frac{7}{2}\delta^3_{HV}\right]\Gamma^{(0)}_{CE} \\ &+ \frac{3}{2}i\frac{k_{CE}}{k_N}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E}q^2}\right) + \frac{4}{3} + \frac{10}{9}\delta^1_{HV} + \delta^2_{HV} + 4\delta^3_{HV}\right]\Gamma^{(0)}_{N}\Bigg\} \\ &+ \mathcal{O}(\epsilon, m),\end{aligned} \tag{9.39b}$$

$$\Gamma^{(1,2A2)}_{CE,CE}(0) = \frac{C_2(A) - 2C_2(F)}{(4\pi)^2}\Bigg\{ \left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + 2 + \frac{1}{3}\delta^1_{HV} + \frac{1}{2}\delta^2_{HV} + \frac{7}{2}\delta^3_{HV}\right]\Gamma^{(0)}_{CE}$$
$$+ \frac{3}{2}i\frac{k_{CE}}{k_N}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + \frac{4}{3} + \frac{10}{9}\delta^1_{HV} + \delta^2_{HV} + 4\delta^3_{HV}\right]\Gamma^{(0)}_{N}\Bigg\}$$
$$+ \mathcal{O}(\epsilon, m), \tag{9.39c}$$

$$\Gamma^{(1,3A1)}_{CE,CE}(0) = \frac{C_2(A)}{(4\pi)^2}\Bigg\{\frac{7}{4}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + \frac{11}{7} + \frac{8}{21}\delta^1_{HV} + \frac{3}{7}\delta^2_{HV} + \frac{24}{7}\delta^3_{HV}\right]\Gamma^{(0)}_{CE}$$
$$+ \frac{3}{4}i\frac{k_{CE}}{k_N}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + \frac{4}{3} - \frac{2}{9}\delta^1_{HV} + 3\delta^3_{HV}\right]\Gamma^{(0)}_{N}\Bigg\}$$
$$+ \mathcal{O}(\epsilon, m), \tag{9.39d}$$

$$\Gamma^{(1,3A2)}_{CE,CE}(0) = \frac{C_2(A)}{(4\pi)^2}\Bigg\{\frac{5}{4}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + \frac{4}{5} + \frac{8}{15}\delta^1_{HV} + \frac{3}{5}\delta^2_{HV} + \frac{18}{5}\delta^3_{HV}\right]\Gamma^{(0)}_{CE}$$
$$- \frac{3}{2}i\frac{k_{CE}}{k_N}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + \frac{4}{3} + \frac{10}{9}\delta^1_{HV} + \delta^2_{HV} + 4\delta^3_{HV}\right]\Gamma^{(0)}_{N}\Bigg\}$$
$$+ \mathcal{O}(\epsilon, m), \tag{9.39e}$$

$$\Gamma^{(1,4A1)}_{CE,CE}(0) = \frac{C_2(A)}{(4\pi)^2}\Bigg\{ - \frac{1}{2}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + 2 + 3\delta^3_{HV}\right]\Gamma^{(0)}_{CE}$$
$$- \frac{3}{2}i\frac{k_{CE}}{k_N}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + \frac{4}{3} + \frac{2}{3}\delta^1_{HV} + \frac{2}{3}\delta^2_{HV} + \frac{11}{3}\delta^3_{HV}\right]\Gamma^{(0)}_{N}\Bigg\}$$
$$+ \mathcal{O}(\epsilon, m), \tag{9.39f}$$

$$\Gamma^{(1,4A2)}_{CE,CE}(0) = \mathcal{O}(m), \tag{9.39g}$$

$$\Gamma^{(1,5A)}_{CE,CE}(0) = \frac{C_2(A)}{(4\pi)^2}\Bigg\{ - \frac{3}{2}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + 2 + 3\delta^3_{HV}\right]\Gamma^{(0)}_{CE}\Bigg\} \tag{9.39h}$$
$$+ \mathcal{O}(\epsilon, m),$$

Summing all contributions on either side, we find the bare correlators:

$$\begin{aligned}\Gamma^{(1)}_{CE,CE}(t) =& \frac{1}{(4\pi)^2}\Bigg\{ \Bigg[(2C_2(F) - 2C_2(A))\log(8\pi t) + \left(\frac{14}{3}C_2(A) - 5C_2(F) + \frac{4}{3}T_F n_f\right)\log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) \\ &- C_2(F)\log\left(\frac{4\pi}{e^{\gamma_E} p^2}\right) + \frac{169}{36}C_2(A) - \frac{13}{2}C_2(F) + \frac{20}{9}T_F n_f\Bigg]\Gamma^{(0)}_{CE} \\ &+ \frac{3}{8}i\frac{k_{CE}}{k_N}\Bigg[(16C_2(F) - 2C_2(A))\log(8\pi t) + (2C_2(A) - 16C_2(F))\log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) \\ &+ C_2(A) - 44C_2(F)\Bigg]\Gamma^{(0)}_{N}\Bigg\} + \mathcal{O}(m, t),\end{aligned} \tag{9.40a}$$

$$\begin{aligned}\Gamma^{(1)}_{CE,CE}(0) =& \frac{1}{(4\pi)^2}\Bigg\{ \Bigg[\left(\frac{14}{3}C_2(A) - 5C_2(F) + \frac{4}{3}T_F n_f\right)\left(\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right)\right) - C_2(F)\left(\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} \cdots}\right.\right. \\ &+ \frac{241}{36}C_2(A) - 5C_2(F) + \frac{20}{9}T_F n_f + \left(2C_2(A) - \frac{4}{3}C_2(F)\right)\delta^1_{HV} + \left(\frac{5}{2}C_2(A) - 2C_2 \cdots\right. \\ &+ \left(\frac{23}{2}C_2(A) - 14C_2(F)\right)\delta^3_{HV}\Bigg]\Gamma^{(0)}_{CE} \\ &+ \frac{3}{8}i\frac{k_{CE}}{k_N}\Bigg[(2C_2(A) - 16C_2(F))\left(\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right)\right) + \frac{8}{3}C_2(A) - \frac{64}{3}C_2(F) \\ &+ \left(\frac{4}{3}C_2(A) - \frac{160}{9}C_2(F)\right)\delta^1_{HV} + \left(\frac{4}{3}C_2(A) - 16C_2(F)\right)\delta^2_{HV} \\ &+ \left(\frac{22}{3}C_2(A) - 64C_2(F)\right)\delta^3_{HV}\Bigg]\Gamma^{(0)}_{N}\Bigg\} + \mathcal{O}(\epsilon, m).\end{aligned} \tag{9.40b}$$

These renormalize as

$$\Gamma^R_{CE,CE}(0) = Z_\psi^{-1} Z_A^{-1} Z_{CE}^{-1}\Gamma_{CE,CE}(0), \tag{9.41}$$

$$\Gamma^R_{CE,CE}(t) = Z_\psi^{-1} Z_A^{-1} Z_\chi^{-1}\Gamma_{CE,CE}(t), \tag{9.42}$$

where we have implicitly renormalized the coupling with

$$g_0^2 = Z_g \mu^{2\epsilon} g^2, \tag{9.43}$$

and the $\overline{\text{MS}}$ Z-factors are

$$Z_g = 1 + \frac{g^2}{(4\pi)^2}\cdot\left[-\frac{11}{3}C_2(A) - \frac{4}{3}T_F n_f\right]\frac{1}{\epsilon} + \mathcal{O}(g^4), \tag{9.44a}$$

$$Z_\xi = 1 + \frac{g^2}{(4\pi)^2}\cdot\left[\frac{13 - 3\xi}{6}C_2(A) + \frac{4}{3}T_F n_f\right]\frac{1}{\epsilon} + \mathcal{O}(g^4), \tag{9.44b}$$

$$Z_\psi = 1 + \frac{g^2}{(4\pi)^2} \cdot [-C_2(F)] \frac{1}{\epsilon} + \mathcal{O}(g^4), \tag{9.44c}$$

$$Z_\chi = 1 + \frac{g^2}{(4\pi)^2} \cdot [-3C_2(F)] \frac{1}{\epsilon} + \mathcal{O}(g^4), \tag{9.44d}$$

$$Z_{CE} = 1 + \frac{g^2}{(4\pi)^2} \cdot [-C_2(A) - C_2(F)] \frac{1}{\epsilon} + \mathcal{O}(g^4), \tag{9.44e}$$

$$Z_A = Z_g^{1/2} Z_\xi^{1/2}. \tag{9.44f}$$

Then, in the Feynman gauge, $\xi = 1$, we have

$$\begin{aligned} c_{CE,CE}(t) = 1 + \frac{g^2}{(4\pi)^2} \cdot \Bigg\{ & [2C_2(F) - 2C_2(A)] \log\left(2e^{\gamma_E}\bar{\mu}^2 t\right) - 2C_2(A) - \frac{3}{2}C_2(F) \\ & - \left(2C_2(A) - \frac{4}{3}C_2(F)\right)\delta^1_{HV} - \left(\frac{5}{2}C_2(A) - 2C_2(F)\right)\delta^2_{HV} \\ & - \left(\frac{23}{2}C_2(A) - 14C_2(F)\right)\delta^3_{HV}\Bigg\} + \mathcal{O}(g^4, m, t) \end{aligned} \tag{9.45}$$

for the self-mixing coefficient and

$$\begin{aligned} c_{CE,N}(t) = 1 + \frac{g^2}{(4\pi)^2} \cdot i\frac{k_{CE}}{k_N}\Bigg\{ & \Bigg[6C_2(F) - \frac{3}{4}C_2(A)\Bigg] \log\left(2e^{\gamma_E}\bar{\mu}^2 t\right) - \frac{5}{8}C_2(A) - \frac{17}{2}C_2(F) \\ & - \left(\frac{1}{2}C_2(A) - \frac{20}{3}C_2(F)\right)\delta^1_{HV} - \left(\frac{1}{2}C_2(A) - 6C_2(F)\right)\delta^2_{HV} \\ & - \left(\frac{11}{4}C_2(A) - 24C_2(F)\right)\delta^3_{HV}\Bigg]\Gamma^{(0)}_N\Bigg\} + \mathcal{O}(\epsilon, m, t) \end{aligned} \tag{9.46}$$

for the coefficient of the purely $SU(N_C)$ nuisance operator. (We have implicitly taken $g_e \to 0$ for this calculation, so the photon term drops out of the covariant derivative.)

9.3 Gluon Chromoelectric Dipole Moment

9.4 Further Results

Part IV

Lattice Applications

Chapter 10

Renormalized Lattice Correlators

10.1 The Quark Chromoelectric Dipole Moment

Chapter 11

Flow-Time Blocking and the Renormalization Group

11.1 Real-Space Treatment

11.2 Perturbative Treatment

If, instead of integrating head-on, we pass the correlation function defined in Eq. **??** to momentum space and inject some momentum q, we may neglect the 4^{th}-component integrals until the end of the analytic portion of this calculation. This has two benefits. First, the oscillating Fourier kernels in Eq. **??** are almost sure to muddle the results of Montecarlo integrators built for perturbation theory, but algorithms for taking numerical Fourier transforms have become very robust and efficient. Waiting to transform – even numerically – until all other integrations are performed may increase the precision of the final results if indeed an analytical result is again unobtainable. The second benefit is the preexisting tools built to handle this sort of integrals, which are designed to work in the full d-, not $(d-1)$-, dimensions. We may recover the time-separated real-space correlators by projecting a fully momentum-space correlator to zero spatial momentum:

$$G_i(x_4;t) = \lim_{p\to 0} \int d^3x \; e^{-ipx} \int_q e^{iqx} \tilde{\Gamma}_{ii}(q;t), \tag{11.1}$$

where

$$\tilde{\Gamma}_{ij}(q;t) = \int d^4x \; e^{-iq(x-y)} \langle \mathcal{O}_i(x;t) \mathcal{O}_j(y;0) \rangle \tag{11.2}$$

The correlator above is fairly manageable. At tree level, the expression is simply

$$\tilde{\Gamma}_{ij}(q;t) = -\operatorname{Tr} \int_p \Gamma_i \tilde{S}^{(0)}(p+q;t) \Gamma_j \tilde{S}^{(0)}(p;t). \tag{11.3}$$

Using standard techniques [**?**], we find that in the massless limit this reduces to a series over scalar integrals with operator-dependent coefficients:

$$\tilde{\Gamma}_{ij}(q;t) = \frac{\dim(F)}{(4\pi)^2}(4\pi\mu^2)^{2-d/2} \sum_{n=0}^{\infty} \left\{ \frac{2^{2n}}{(2n)!} X_n^{ij}(q) \int_0^\infty dz e^{-q^2(t+z)} \frac{(t+z)^{2n}}{(2t+z)^{n+d/2}} \right. \\ \left. - \frac{2^{2n+1}}{(2n+1)!} Y_n^{ij}(q) \int_0^\infty dz e^{-q^2(t+z)} \frac{(t+z)^{2n+1}}{(2t+z)^{n+d/2}} \right\}, \tag{11.4}$$

where

$$
\begin{aligned}
X_n^{SS}(q) &= -X_n^{PP}(q) = 4(2n-1)!!(d+2n)(q^2)^n \\
X_n^{VV}(q) &= -X_n^{AA}(q) = -4(2n-1)!!\left((d+2n-2)\delta_{\mu\nu} - 4n\frac{q_\mu q_\nu}{q^2}\right)(q^2)^n \\
X_n^{TT}(q) &= \\
Y_n^{SS}(q) &= -Y_n^{PP}(q) = 4(2n+1)!!(q^2)^{n+1} \\
Y_n^{VV}(q) &= -Y_n^{AA}(q) = -4(2n+1)!!\left(\delta_{\mu\nu} - 2\frac{q_\mu q_\nu}{q^2}\right)(q^2)^{n+1} \\
Y_n^{TT}(q) &=
\end{aligned}
\tag{11.6}
$$

Now for demonstration, let us specialize to the scalar correlator, $\tilde{\Gamma}_{SS}(q;t)$. Moving to four dimensions, summing over n, and integrating in z, we find

$$
\tilde{\Gamma}_{SS}(q;t) = \tilde{\Gamma}_{SS}^{(0)}(t) \cdot \tilde{f}(q^2 t), \tag{11.7}
$$

where $\tilde{\Gamma}_{SS}^{(0)}(t)$ is the tree-level result integrated in d^4x (*v.i.* Eq. **??**), and

$$
\frac{1}{2}\tilde{f}(z) = \frac{z}{2}E_1\left(\frac{z}{2}\right) - zE_1(z) - \frac{z-1}{z}\left(e^{-z/2} - e^{-z}\right). \tag{11.8}
$$

We now implement the Fourier transform as in Eq. 11.1:

$$
\begin{aligned}
G_S(x_4;t) = &\lim_{p\to 0}\int d^3x e^{-ipx}\int_q e^{iqx}\tilde{\Gamma}_{SS}(q;t) \\
&= \lim_{p\to 0}\int_{q_4} e^{-i(p_4-q_4)x_4}\int_{\vec{q}}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\tilde{\Gamma}_{SS}(q;t) \\
&= \lim_{p\to 0}\int_{q_4} e^{-i(p_4-q_4)x_4}\tilde{\Gamma}_{SS}((\vec{p},q_4);t) \\
&= \int_{q_4} e^{iq_4x_4}\tilde{\Gamma}_{SS}((\vec{0},q_4);t) \\
&= \tilde{\Gamma}_{SS}^{(0)}(t)\int_{q_4} e^{iq_4x_4}\tilde{f}(q_4^2 t).
\end{aligned}
\tag{11.9}
$$

There are three unique inverse transforms to consider here. Let us expand $f(q^2t)$:

$$
\tilde{f}(q^2t) = 2\left\{\frac{q^2t}{2}E_1\left(\frac{q^2t}{2}\right) - q^2tE_1\left(q^2t\right) + \frac{2}{q^2t}e^{-\frac{q^2t}{2}} - \frac{1}{q^2t}e^{-q^2t} - e^{-\frac{q^2t}{2}} + e^{-q^2t} - \frac{1}{2}\frac{2}{q^2t}e^{-q^2t}\right\} \tag{11.10}
$$

Defining the three functions

$$
\begin{aligned}
f_1(x;\tau) &= \int \frac{dq}{2\pi}e^{iqx}q^2\tau E_1(q^2\tau), \\
f_2(x;\tau) &= \int \frac{dq}{2\pi}e^{iqx}\frac{e^{-q^2\tau}}{q^2\tau}, \\
f_3(x;\tau) &= \int \frac{dq}{2\pi}e^{iqx}e^{-q^2\tau},
\end{aligned}
\tag{11.11}
$$

we see that the problem is reduced to a linear combination of these functions at different points z:

$$f(x_4;t) = \int_{q_4} e^{iq_4 x_4} \tilde{f}(q_4^2 t) = -2\left[f_1(x_4;\tau) + f_2(x_4;\tau) - f_3(x_4;\tau)\right]\Big|_{\tau=t/2}^{t} - f_2(x_4;\tau)\Big|_{\tau=t/2}. \tag{11.12}$$

Computing f_1 is fairly straightforward. We first replace the factor of q^2 in the integrand by a double derivative with respect to x and recast the exponential integral E_1 in integral from:

$$f_1(x;\tau) = \int_q e^{iqx} q^2 \tau E_1(q^2\tau) = -\tau\partial_x^2 \int_q e^{iqx} E_1(q^2\tau) = -\frac{\tau}{2\pi}\partial_x^2 \int_1^\infty \frac{d\alpha}{\alpha} \int_{-\infty}^{\infty} dq\; e^{iqx} e^{-q^2\tau\alpha} \tag{11.13}$$

The integral with respect to q is calculable by contour integration. We first make the change of variables $z = q - \frac{ix}{2\tau\alpha}$ and pull the implicit limit out of the improper integral:

$$f_1(x;\tau) = -\frac{\tau}{2\pi}\partial_x^2 \int_1^\infty \frac{d\alpha}{\alpha} e^{-\frac{x^2}{4\tau\alpha}} \lim_{T\to\infty} \int_{-T-i\delta}^{T-i\delta} dz\; e^{-(\tau\alpha)z^2}, \tag{11.14}$$

where $\delta = x/2\tau\alpha$. Referring to Fig. 11.1, we notice that the path of integration in z is exactly the curve $\mathcal{C}_1^\pm$ for $\delta = \pm|\delta|$. Then we may write

$$f_1(x;\tau) = -\frac{\tau}{2\pi}\partial_x^2 \int_1^\infty \frac{d\alpha}{\alpha} e^{-\frac{x^2}{4\tau\alpha}} \lim_{T\to\infty} \left(\oint_{\mathcal{C}^\pm} dz - \int_{\mathcal{C}_2^\pm} dz - \int_{\mathcal{C}_3^\pm} dz - \int_{\mathcal{C}_4^\pm} dz \right) e^{-(\tau\alpha)z^2}. \tag{11.15}$$

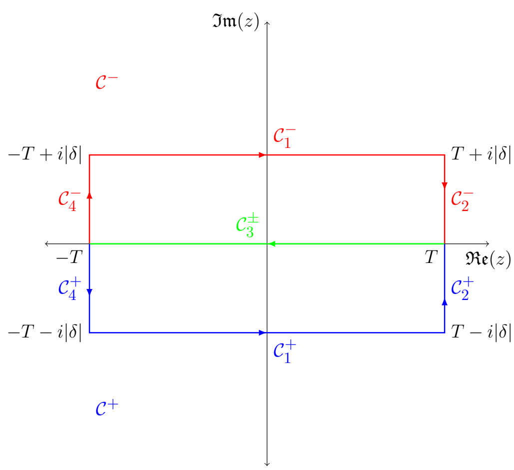


Figure 11.1

Since the integrand is an entire function, the contour integral vanishes. Changing variables, we have

$$f_1(x;\tau) = -\frac{\tau}{2\pi}\partial_x^2 \int_1^\infty \frac{d\alpha}{\alpha} e^{-\frac{x^2}{4\tau\alpha}} \lim_{T\to\infty} \Bigg\{ \pm 2\pi i \cdot 0 - \int_{|\delta|}^0 (\mp i) dy\ e^{-(T\mp iy)^2\tau\alpha} - \int_T^{-T} dx\ e^{-x^2\tau\alpha} - \int_0^{|\delta|} (\mp i) dy\ e^{-(T\pm iy)^2\tau\alpha} \Bigg\}. \tag{11.16}$$

The integrals over $\mathcal{C}_2^\pm$ and $\mathcal{C}_4^\pm$ combine so that

$$f_1(x;\tau) = -\frac{\tau}{2\pi}\partial_x^2 \int_1^\infty \frac{d\alpha}{\alpha} e^{-\frac{x^2}{4\tau\alpha}} \lim_{T\to\infty} \Bigg\{ \int_{-T}^T dx\, e^{-x^2\tau\alpha} + 2e^{-T^2\tau\alpha} \int_0^{|\delta|} dy\, e^{y^2\tau\alpha} \sin(2T\tau\alpha y) \Bigg\}. \tag{11.17}$$

The second integral in brackets is suppressed by a factor of $e^{-T^2\tau\alpha}$ with $\tau\alpha \geq 0$, so it vanishes with large T (Incidentally, the integral alone vanishes in this limit as well). We are now left with a simple Gaussian integral, which evaluates to

$$f_1(x;\tau) = -\frac{\tau}{2\pi}\partial_x^2 \int_1^\infty \frac{d\alpha}{\alpha} e^{-\frac{x^2}{4\tau\alpha}} \sqrt{\frac{\pi}{\tau\alpha}}. \tag{11.18}$$

This result also gives us f_3, which equals the integrand above divided by $(-\tau)$ and evaluated at $\alpha = 1$:

$$f_3(x;\tau) = \frac{e^{-\frac{x^2}{4\tau}}}{2\sqrt{\pi\tau}} \tag{11.19}$$

Returning to f_1, one more change of variables, $\beta = 1/\sqrt{\alpha}$, gives us

$$f_1(x;\tau) = -\sqrt{\frac{\tau}{\pi}}\partial_x^2 \int_0^1 d\beta\ e^{-\beta^2\frac{x^2}{4\tau}} = -\partial_x^2 \frac{\tau}{x} \operatorname{erf}\left(\frac{x}{2\sqrt{\tau}}\right) \tag{11.20}$$

This derivative evaluates to

$$f_1(x;\tau) = \frac{1}{4\sqrt{\pi\tau}} \left[2e^{-\epsilon^2}\left(1 + \frac{1}{\epsilon^2}\right) - \frac{\sqrt{\pi}}{\epsilon^3} \operatorname{erf}(\epsilon) \right], \tag{11.21}$$

where $\epsilon = x/2\sqrt{\tau}$. Notably, this expression is clearly even in ϵ (therefore x_4), as it should be.

We now move on to f_2, which we begin by writing as a convolution:

$$f_2(x,\tau) = \int \frac{dq}{2\pi} e^{iqx} \frac{e^{-q^2\tau}}{q^2\tau} = \int_{-\infty}^\infty dz \int_{-\infty}^\infty \frac{dq}{2\pi} \frac{e^{iqz}}{q^2\tau} \int_{-\infty}^\infty \frac{dp}{2\pi} e^{ip(x-z)} e^{-p^2 t} = \int_{-\infty}^\infty dz \frac{e^{-\frac{(x-z)^2}{4\tau}}}{2\sqrt{\pi\tau}} \int_{-\infty}^\infty \frac{dq}{2\pi} \frac{e^{iqz}}{q^2\tau} \tag{11.22}$$

where in the last equality we have inserted $f_3(x-z;\tau)$. The integral in q must be interpreted by meromorphic continuation. Interestingly, this requirement anticipates the result: an even function that is analytic on the punctured line. We define the Hadamard finite-part integral as

$$\int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqz}}{q^2\tau} \equiv \mathcal{H} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqz}}{q^2\tau} = \lim_{a\to 0^+} \frac{1}{2\pi\tau} \frac{d}{da} \left(\text{p.v.} \int_{-\infty}^{\infty} \frac{d}{q} \frac{e^{iqz}}{q-a} \right). \tag{11.23}$$

This may be rewritten

$$\mathcal{H} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqz}}{q^2\tau} = \frac{1}{2\pi\tau} \lim_{a,\epsilon\to 0^+} \left\{ \int_{-\infty}^{\infty} dq \frac{(q-a)^2 e^{iqz}}{((q-a)^2+\epsilon^2)^2} - \frac{\pi}{2\epsilon} e^{iaz} \right\}. \tag{11.24}$$

This expression makes clear the double poles at $a \pm i\epsilon$. Integrating over the semicircular contour $\mathcal{C}^{\pm}$ for $z \gtrless 0$ as in Fig. 11.2, we write

$$\mathcal{H} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqz}}{q^2\tau} = \frac{1}{2\pi\tau} \lim_{a,\epsilon\to 0^+} \lim_{R\to\infty} \left\{ \left(\oint_{\mathcal{C}^{\pm}} dq - \int_{\mathcal{C}_R^{\pm}} dq \right) \frac{(q-a)^2 e^{iqz}}{((q-a)^2+\epsilon^2)^2} - \frac{\pi}{2\epsilon} e^{iaz} \right\}. \tag{11.25}$$

The integral over the curve $\mathcal{C}_R^{\pm}$ vanishes by Jordan's Lemma, and the residue theorem gives us

$$\mathcal{H} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqz}}{q^2\tau} = \frac{1}{2\pi\tau} \lim_{a,\epsilon\to 0^+} \left\{ (\pm 2\pi i) \lim_{q\to a\pm i\epsilon} \partial_q \frac{(q-a)^2 e^{iqz}}{(q-(a\mp i\epsilon))^2} - \frac{\pi}{2\epsilon} e^{iaz} \right\}. \tag{11.26}$$

The limit term becomes

$$\pm 2\pi i \lim_{q\to a\pm i\epsilon} \partial_q \frac{(q-a)^2 e^{iqz}}{(q-(a\mp i\epsilon))^2} = \frac{\pi}{2} e^{iaz} e^{\mp z\epsilon} \left(\frac{1}{\epsilon} \mp a - i\epsilon \right) = \frac{\pi}{2} e^{iaz} \left[\frac{1}{\epsilon} \mp (z+a) \right] + \mathcal{O}(\epsilon), \tag{11.27}$$

and we see that the pole at $\epsilon = 0$ cancels readily with this definition of the Hadamard integral, so that

$$\mathcal{H} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqz}}{q^2\tau} = \frac{1}{2\pi\tau} \lim_{a,\epsilon\to 0^+} \frac{\pi e^{i\alpha z}}{2\epsilon} \left(e^{\mp z\epsilon}(1 \mp z\epsilon) - 1 \right) = \mp \frac{z}{2\tau} \lim_{a\to 0^+} e^{iaz} = \mp \frac{z}{2\tau}; \qquad z \gtrless 0. \tag{11.28}$$

This reduces simply to

$$\mathcal{H} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{e^{iqz}}{q^2\tau} = -\frac{|z|}{2\tau}. \tag{11.29}$$

Returning to Eq. 11.22

$$f_2(x,\tau) = -\frac{1}{4\pi^{1/2}\tau^{3/2}} \int_{-\infty}^{\infty} dz\, |z| e^{-\frac{(x-z)^2}{4\tau}} = -\frac{1}{4\pi^{1/2}\tau^{3/2}} \left\{ -\int_{-\infty}^{0} dz\ z e^{-\frac{(x-z)^2}{4\tau}} + \int_{0}^{\infty} dz\ z e^{-\frac{(x-z)^2}{4\tau}} \right\}. \tag{11.30}$$

We can change the sign under the first integral and combine to find

$$f_2(x,\tau) = -\frac{1}{2\pi^{1/2}\tau^{3/2}} \int_0^{\infty} dz\ z e^{-\frac{x^2+z^2}{4\tau}} \cosh\left(\frac{xz}{2\tau} \right). \tag{11.31}$$

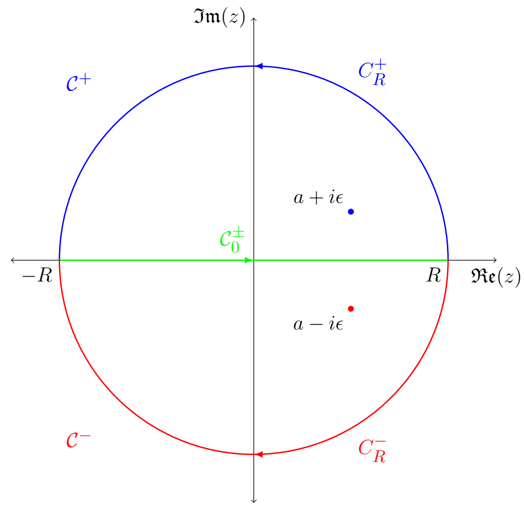


Figure 11.2

Expanding the cosh and then integrating in z,

$$f_2(x,\tau) = -\frac{e^{-\epsilon^2}}{\sqrt{\pi\tau}} \sum_{n=0}^{\infty} \frac{(\sqrt{2}\epsilon)^{2n}}{(2n-1)!!}. \tag{11.32}$$

Finally, we pull out the leading order and shift the summation index by one to find the Taylor series for the error function:

$$f_2(x,\tau) = -\frac{e^{-\epsilon^2}}{\sqrt{\pi\tau}} \left\{ 1 + \sqrt{2}\epsilon \sum_{n=0}^{\infty} \frac{(\sqrt{2}\epsilon)^{2n+1}}{(2n+1)!!} \right\} = -\frac{1}{\sqrt{\pi\tau}} \left\{ e^{-\epsilon^2} + \sqrt{\pi}\epsilon \ \operatorname{erf}(\epsilon) \right\} \tag{11.33}$$

We may now write the final result for $G_S(x_4;t)$:

$$G_S(x_4;t) = \tilde{\Gamma}^{(0)}_{SS}(t) \left\{ \frac{1}{4\sqrt{t}\epsilon^3} \left[\left(8\epsilon^4 + 2\right) \operatorname{erf}(\epsilon) - \left(8\epsilon^4 + 1\right) \operatorname{erf}\left(\sqrt{2}\epsilon\right) \right] - \frac{1}{\sqrt{\pi t}} \left[\left(1 - \frac{1}{2\epsilon^2}\right) \left(\sqrt{2} e^{-2\epsilon^2} - 2e^{-} \right.\right.\right. \tag{11.34}$$

APPENDICES

Appendix A

Conventions

Appendix B

Feynman Rules

Appendix C

Perturbation Theory Techniques

Appendix D

Combinatorial Tensor Decomposition

In the perturbation theory associated with the Yang-Mills gradient flow, one regularly encounters integrals of the form

$$I_{ijk}^{abc}(p,\mu;t) = \int_q \frac{e^{-ap^2} e^{-bq^2} e^{-c(p+q)^2}}{\left(p^2\right)^i \left(q^2\right)^j \left((p+q)^2\right)^k}; \qquad a,b,c,i,j,k \in \mathbb{R}, \tag{D.1}$$

where we define the shorthand

$$\int_q \equiv \mu^{4-d} \int_{\mathbb{R}^d} \frac{d^d q}{(2\pi)^d}. \tag{D.2}$$

The energy scale μ and the dimension $d = 4 - 2\epsilon$ are included to prepare the integral for dimensional regularization and renormalization by construction. We will, for full generality, solve all integrals in d-dimensions, which will allow the reader to modify the integrals without worrying about regulators.

D.1 Standard Integrals in Dimensional Regularization

Without the gradient flow, loop integrals are relatively simple to generalize to d-dimensions, where they generally take the form

$$I_{\mu_I}^{n_I}(p_I; m_I) = \int_q \frac{q_{\mu_1} \cdots q_{\mu_n}}{\prod_{i=1}^N \left(r_i^2 + m_i^2\right)^{n_i}} \tag{D.3}$$

with muti-index $I = \{1, \ldots, N\}$, where the product in the denominator runs over all propagators in the loop with their respective masses and momenta indexed by i, and the vectors in the numerator are the result of n derivative couplings (non-scalar vertices). Each r_i in the denominator has the form $r_i = q + s_i$, where $s_i = p_1 + \cdots + p_i$ for external momenta $\{p_i\}_{i=1}^N$. The dimension of the integral is undetermined and generically non-integral, so we cannot integrate over each component directly. If the integrand is spherically symmetric, however, we may transform to spherical coordinates, where the $(d-1)$-dimensional spherical shell is readily integrated out, as follows. First we extract the solid angle:

$$\int_q f(q^2) = \frac{\mu^{4-d}}{(2\pi)^d} \int_\Omega d\Omega \int_0^\infty r^{d-1} dr\ f(r^2), \tag{D.4}$$

where

$$d\Omega = \prod_{k=1}^{d-1} \sin^{d-k-1}(\phi_k) d\phi_k. \tag{D.5}$$

and where the angular domain Ω is defined by

$$\begin{aligned} \phi_k &\in [0, \pi); \quad k < d-1 \\ \phi_k &\in [0, 2\pi); \quad k = d-1. \end{aligned} \tag{D.6}$$

Symmetry allows us to write

$$\int_0^{\pi} d\phi_k \, \sin^{d-k-1}(\phi_k) = 2 \int_0^{\pi/2} d\phi_k \, \sin^{d-k-1}(\phi_k); \quad k < d-1 \tag{D.7}$$

and

$$\int_0^{2\pi} d\phi_k \, \sin^{d-k-1}(\phi_k) = 4 \int_0^{\pi/2} d\phi_k; \quad k = d-1, \tag{D.8}$$

so that

$$\begin{aligned} \int_\Omega d\Omega =& 2 \prod_{k=1}^{d-1} \left[2 \int_0^{\pi/2} d\phi_k \, \sin^{d-k-1}(\phi_k) \right] \\ =& 2 \prod_{k=1}^{d-1} B\left(\frac{d-k}{2}, \frac{1}{2} \right) \\ =& 2\Gamma^{d-1}\left(\frac{1}{2} \right) \prod_{k=1}^{d-1} \Gamma\left(\frac{d-k}{2} \right) \Big/ \Gamma\left(\frac{d-k+1}{2} \right). \end{aligned} \tag{D.9}$$

The numerator of each factor cancels the denominator of the next, and we are left with

$$\int_\Omega d\Omega = 2\pi^{\frac{d-1}{2}} \frac{\Gamma(1/2)}{\Gamma(d/2)} = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{D.10}$$

Thus

$$\int_q f(q^2) = \frac{2\mu^{4-d}}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty r^{d-1} dr \, f(r^2). \tag{D.11}$$

If the integrand is not even, we must transform it to a spherical form, the standard for which is Feynman parameterization. The identity

$$\frac{1}{\prod_{i=1}^N \left(r_i^2 + m_i^2\right)^{n_i}} = \frac{1}{B(n_1, \ldots n_N)} \int_0^\infty dz_1 \cdots \int_0^\infty dz_N \frac{\delta\left(1 - \sum_{i=1}^N z_i\right) \prod_{i=1}^N z_i^{n_i - 1}}{\left[\sum_{i=1}^N z_i \left(r_i^2 + m_i^2\right)\right]^{\sum_{i=1}^N n_i}} \tag{D.12}$$

allows the denominator to be expressed as a sum, so that we can complete the square in the momentum of integration q:

$$
\begin{aligned}
I^{n_I}_{\mu_I}(p_I;m_I) &= \frac{1}{B(n_1,\dots n_N)}\int_0^\infty \prod_{i=1}^N \left(z_i^{n_i-1}dz_i\right)\int_q \frac{\delta\left(1-\sum_{i=1}^N z_i\right)}{\left[\sum_{i=1}^N z_i\left((q+s_i)^2+m_i^2\right)\right]^{\sum_{i=1}^N n_i}} q_{\mu_1}\cdots q_{\mu_n} \\
&= \frac{1}{B(n_1,\dots n_N)}\int_0^\infty \prod_{i=1}^N \left(z_i^{n_i-1}dz_i\right)\int_q \frac{\delta\left(1-\sum_{i=1}^N z_i\right)}{\left[(q+Q)^2+\Delta\right]^{\sum_{i=1}^N n_i}} q_{\mu_1}\cdots q_{\mu_n},
\end{aligned}
\tag{D.13}
$$

where

$$
\Delta = \sum_{i=1}^N z_i(s_i^2+m_i^2) - Q^2, \tag{D.14}
$$

and

$$
Q_\mu = \sum_{i=1}^N z_i (s_i)_\mu. \tag{D.15}
$$

Under the change of variables $k_\mu = q_\mu + Q_\mu$, we have,

$$
I^{n_I}_{\mu_I}(p_I;m_I) = \frac{1}{B(n_1,\dots n_N)}\int_0^\infty \prod_{i=1}^N \left(z_i^{n_i-1}dz_i\right)\int_k \frac{\delta\left(1-\sum_{i=1}^N z_i\right)}{\left[k^2+\Delta\right]^{\sum_{i=1}^N n_i}} (k-Q)_{\mu_1}\cdots (k-Q)_{\mu_n}, \tag{D.16}
$$

and the evenness or oddness of the integrand is more obvious. The product of vectors $(k-Q)_{\mu_i}$ is a polynomial in k, so the even-degree terms will survive integration, and the odd terms will vanish. Since the fraction above is even, the momentum integral is a sum over integrals of the form

$$
\int_q f(q^2) q_{\mu_1}\cdots q_{\mu_{2n}}, \tag{D.17}
$$

for some n. The $2n$-fold product ensures that the integral does not trivially vanish, but we now must discern the tensor structure.

D.2 Reduction of Tensor Integrals

The solution of the integral must have the same symmetry as the integrand, so it must be proportional to some tensor with such symmetry:

$$
\int_q f(q^2)\prod_{m=1}^{2n} q_{\mu_m} = A\cdot T_{\mu_1\cdots\mu_{2n}}. \tag{D.18}
$$

Since the product is commutative, it is entirely symmetric with respect to any permutation of the $2n$ indices μ_m. The only tensor with such a symmetry is the symmetrized sum of products of n metric tensors over all unordered partitions of the $2n$ indices into n pairs. Let $\sigma_r(s)$ denote the r^{th} permutation on the index s of this form. Note that under these restrictions, the following partitions are all equivalent:

$$\{1,2\},\{3,4\},\{5,6\}\{2,1\},\{3,4\},\{5,6\}\{3,4\},\{1,2\},\{5,6\} \tag{D.19}$$

We must first count the number of ways we may group $2n$ indices into n pairs. Choosing an index generically, there are $2n-1$ remaining indices available for pairing. Continuing in this manner, there are $2n-2$ indices we may choose to begin the second pair, with $2n-3$ partners remaining, and so on to $(2n)!$. Since the ordering of the pairs doesn't matter, we divide by $n!$. Moreover, each pair is itself unordered with respect to its two elements, so we divide again by 2^n. There are, then, $\frac{(2n)!}{2^n n!} = (2n-1)!!$ distributions of the indices, and the sum over each permutation $\sigma_r(s)$ runs from $r = 1$ to $r = (2n-1)!!$ with $s \in [1, n] \cap \mathbb{N}$, so

$$T_{\mu_1 \cdots \mu_{2n}} = \sum_{r=1}^{(2n-1)!!} \prod_{s=1}^{n} g_{\mu_{\sigma_r(2s-1)} \mu_{\sigma_r(2s)}} \tag{D.20}$$

To find the constant of proportionality A, contract both sides of equation (D.18) with any term of the sum over permutations; any term may be chosen due to its symmetrical construction. Without loss of generality, we make the natural choice $g_{\mu_1\mu_2} \cdots g_{\mu_{2n-1}\mu_{2n}}$:

$$\prod_{k=1}^{n} g_{\mu_{2k-1}\mu_{2k}} \int_q f(q^2) \prod_{m=1}^{2n} q_{\mu_m} = A \prod_{k=1}^{n} g_{\mu_{2k-1}\mu_{2k}} \sum_{r=1}^{(2n-1)!!} \prod_{s=1}^{n} g_{\mu_{\sigma_r(2s-1)} \mu_{\sigma_r(2s)}}. \tag{D.21}$$

Commutativity and associativity under addition allow us to rearrange the products and contract all indices first, resulting in a scalar expression. On the left, the components of the momentum q are simply paired into a product of n squares, leaving $(q^2)^n$ in place of the integrand's product. The right side is far less trivial, and it will require a bit of care. Before we tackle this problem, however, we note that the integral has been indeed reduced to a scalar, and we are left with a combinatorial problem on the right-hand side:

$$\frac{1}{A} \int_q f(q^2) \cdot (q^2)^n = \prod_{k=1}^{n} g_{\mu_{2k-1}\mu_{2k}} \sum_{r=1}^{(2n-1)!!} \prod_{s=1}^{n} g_{\mu_{\sigma_r(2s-1)} \mu_{\sigma_r(2s)}}. \tag{D.22}$$

D.3 Normalizing the Totally Symmetric Tensor with Graphs

The product on the right-hand side of equation (D.22),

$$s_n = \prod_{k=1}^{n} g_{\mu_{2k-1}\mu_{2k}} \sum_{r=1}^{(2n-1)!!} \prod_{s=1}^{n} g_{\mu_{\sigma_r(2s-1)} \mu_{\sigma_r(2s)}}, \tag{D.23}$$

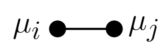
where we have introduced the shorthand s_n, is most easily illustrated by examining the $n = 2$ case, where it reads

$$s_2 = \prod_{k=1}^{2} g_{\mu_{2k-1}\mu_{2k}} \sum_{r=1}^{3} \prod_{s=1}^{2} g_{\mu_{\sigma_r(2s-1)}\mu_{\sigma_r(2s)}} = g_{\mu\nu} g_{\rho\sigma} \cdot \left(g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho} \right) . \tag{D.24}$$

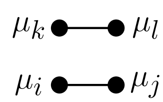
Contracting all indices gives $s_2 = d^2 + d + d$, following the order of the parenthetical term. We notice that only the first term shares the ordering of the indices $g_{\mu\nu} g_{\rho\sigma}$, while the other two do not. Since the first term shares this pairing, there is an n-fold product over traces $g_{\mu\nu} g_{\mu\nu}$, each of which evaluates to the value of the dimension d. The second and third terms differ by a transposition, so the factor outside the parentheses serves to connect the permuted indices, *e.g.*:

$$g_{\mu\nu} g_{\rho\sigma} \cdot g_{\mu\rho} g_{\nu\sigma} = g_{\mu\sigma} g_{\mu\sigma} = g_{\mu\mu} = d, \tag{D.25}$$

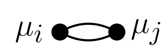
but in doing so, we lose two powers of the metric tensor, so the result will be correspondingly reduced by a trace, or, in other words, one power of the dimension. Since in the preceding case n was very small, there is no need for more advanced machinery. Recall, however, that the number of terms in the parentheses will grow as $(2n-1)!!$. Even for the $n = 3$ case, there are 15 terms, and the result is not trivial. At $n = 4, 5, \ldots$, there are $105, 945, \ldots$ terms, and the number of contractions becomes intractable. Fortunately, this problem is mapped very cleanly to graphs. Let each of the $2n$ indices μ_i represent a vertex on a graph $\mathcal{G}_{2n}^{I}$, where the multi-index $I = \{1, 2, \ldots, 2n\}$ represents the ordered set of indices being mapped to the graph's vertices. Then let each metric tensor represent an edge connecting the vertices corresponding to its indices. Since each index appears once and only once in each term, and since the metric tensor connects only two indices, we have the mapping

$$g_{\mu_i\mu_j} \mapsto$$  . (D.26)

Since each metric tensor only connects two points, we can write the n-fold product of (uncontracted) metric tensors as a (disjoint) graph union:

$$g_{\mu_i\mu_j} g_{\mu_k\mu_l} \mapsto \mathcal{G}_2^{ij} \oplus \mathcal{G}_2^{kl} =$$  . (D.27)

If we have a product of metric tensors with repeated indices, then we take a simple graph union:

$$g_{\mu_i\mu_j} g_{\mu_i\mu_j} \mapsto \mathcal{G}_2^{ij} \cup \mathcal{G}_2^{ij} =$$  . (D.28)

When a cycle appears as above, we recognize the trace of a metric tensor; since every edge corresponds to a metric tensor, and there are no 1-valent vertices, every index is contracted until we are left with a trace over a single metric tensor. These cycles, then, map back to powers of the dimension d, and a graph with k cycles corresponds to d^k. Then we have a correspondence:

$$\prod_{s=1}^{n} g_{\mu_{\sigma_r(2s-1)}\mu_{\sigma_r(2s)}} \sim \mathcal{G}_{2n}^{I_r}, \tag{D.29}$$

where the multi-index I_r is defined by $I_r = \{\sigma_r(1), \dots, \sigma_r(2n)\}$, and edges are meant to exist between every two vertices as they are ordered in I_r. These graphs are 1-regular, since there are no repeated indices in the product of metric tensors, but there is a metric tensor (edge) pairing each index (vertex) to one other. We define $\mathcal{G}_{2n}^{I_i I_j} = \mathcal{G}_{2n}^{I_i} \cup \mathcal{G}_{2n}^{I_j}$ to be the 2-regular union of 1-regular graphs with edges defined by the i^{th} and j^{th} permutations on the indices. Each term in the sum s_n then maps to a graph $\mathcal{G}_{2n}^{I_i I_j}$ for some $i, j \in \{1, 2, \dots, (2n-1)!!\}$. Since $\mathcal{G}_{2n}^{I_i I_j}$ is 2-regular, it must be a union of cycles, each of which evaluates to a power of d. Summarily: each term $\left(g_{\mu_1\mu_2} \cdots g_{\mu_{2n-1}\mu_{2n}}\right)\left(g_{\mu_{\sigma_r(1)}\mu_{\sigma_r(2)}} \cdots g_{\mu_{\sigma_r(2n-1)}\mu_{\sigma_r(2n)}}\right)$ in s_n maps to a 2-regular graph $\mathcal{G}_{2n}^{I I_r}$ which contains k cycles, and maps back to d^k. Thus s_n has the form

$$s_n = \sum_{k=1}^{n} G(n,k) d^k, \tag{D.30}$$

where $G(n,k)$ is the number of 2-regular graphs containing the 1-regular subgraph $\mathcal{G}_{2n}^{I}$, which decompose into k cycles. Note that this number is invariant under the choice of multi-index I. We are now left to the problem of counting these graphs. Fortunately, we may construct them recursively. Consider any such 2-regular graph on $2n$ indices. If we wish to create another 2-regular graph on $2n+2$ vertices, we may add the vertices μ_{2n+1} and μ_{2n+2} in $d + 2n$ ways, as follows. First, note that there must be an edge between μ_{2n+1} and μ_{2n+2}. This comes from the restriction that we must recover the 1-regular graph $\mathcal{G}_{2n+2}^{I}$ as a subgraph, and such a subgraph must contain every edge from μ_{2i+1} and μ_{2i+2} for all $i \in \{0, 1, \dots, n-1\}$ and no more. For the same reason, every graph we create by adding μ_{2n+1} and μ_{2n+2} also must contain all edges of this form. Therefore, we can add the two new vertices as a disjoint 2-cycle, or we may break any existing cycle and insert the new vertices, increasing the size of the cycle by two. Thus the first case corresponds to a disjoint union and its natural mapping back to a polynomial in d:

$$\mathcal{G}_2^{\{\mu_{2n+1},\mu_{2n+2}\}\{\mu_{2n+1},\mu_{2n+2}\}} \oplus \left(\bigcup_{r=1}^{(2n-1)!!} \mathcal{G}_{2n}^{I I_r} \right) \mapsto d \cdot s_n. \tag{D.31}$$

Onto the latter case, since each graph is 2-regular, then $|E| = |V|$ necessarily. Then, for the graph on $2n$ vertices, we may cut each of the $2n$ edges and insert the new vertices. We can insert this new edge in 2 ways for each cut, but we must retain the subgraph $\mathcal{G}_{2n}^{I}$, so half of the cuts produce unusable graphs, and we must divide by two. Then we have $2n$ graphs for each of the graphs in s_n, which adds $2n \cdot s_n$ to our total. This gives us a recursion:

$$s_{n+1} = (d + 2n) \cdot s_n. \tag{D.32}$$

Finally, induction on n gives us

$$s_n = d(d+2)(d+4) \cdots (d + 2(n-1)) = (d)_{n,2}, \tag{D.33}$$

where

$$(d)_{n,2} = \frac{2^n \Gamma(d/2 + n)}{\Gamma(d/2)} \tag{D.34}$$

is the Pochhammer 2-symbol. Using the identity

$$(x)_{n,k} = k^n \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \left(\frac{x}{k}\right)^j, \tag{D.35}$$

we find that

$$G(n,k) = 2^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}. \tag{D.36}$$

In fact, there is a reason for the Stirling numbers of the first kind to appear, and we present an alternative proof of equation (D.33) via explicit permutations on the indices in Appendix D.4. Finally, we have

$$\int_q f(q^2) \prod_{m=1}^{2n} q_{\mu_m} = \frac{1}{(d)_{n,2}} T_{\mu_1 \cdots \mu_{2n}} \cdot \int_q f(q^2) \cdot (q^2)^n. \tag{D.37}$$

The case for $n = 3$ is illustrated in Appendix D.5

D.4 Normalizing the Symmetric Tensor (Alternative Method)

Whichever term has the identical arrangement of indices scompared to the term with which we chose to contract will evaluate to d^n, where d is the dimension, since the result is a product of n traced metric tensors, each equal to d by definition. For each remaining term in the sum over distributions, we wish to find the number of interchanges of indices which will return the ordering to the arbitrary arrangement with which we are contracting; in our case, we want to return each permutation to the natural numerical order $(1,2); (3,4); \ldots; (2n-1, 2n)$. Begin by fixing the first element of the chosen permutation (the odd numbers in our scenario); this leaves n free indices, so we consider the permutation group S_n. We now decompose each element into k disjoint cycles, where k ranges from 1 to n. These may be counted using the unsigned Stirling numbers of the first kind $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = |S_1(n,k)|$; specifically, there are $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ elements of S_n which may be decomposed as the composition of k disjoint cycles. The unsigned Stirling numbers of the first kind are recursively defined as

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = n \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix} \tag{D.38}$$

for $k > 0$ with the initial conditions that

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ n \end{bmatrix} = 0 \text{ and } \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1 \tag{D.39}$$

for $n > 0$. These may be further decomposed into $n - k$ transpositions. Transpositions are functionally equivalent to contraction with a metric tensor indexed by the two indices to be transposed. For each contraction, the exponent of d will be reduced by one, since we have one fewer square of a metric tensor. Since there are $n - k$ transpositions for some term, we have $d^{n-(n-k)} = d^k$. We now consider the weight factor for each k, since we have obviously ignored many (namely $(2n-1)!!-n!$) distributions of pairs by fixing the first index of each pair. The remaining distributions may be constructed by transposing the indices for the pairs. The term with which we choose to contract is insensitive to such transpositions, so the powers of d on the left should be as well. We are simply counting multiplicities of each power of d. For each k of our fixed-index permutations under S_n, we can construct 2^{n-k} permutations with the pairwise transposition symmetry of our chosen term, since each permutation has been decomposed into $n-k$ transpositions. Thus for each k we have $2^{n-k}\left[{n \atop k}\right]$ terms which evaluate to d^k. We now sum over the permutations, which has been shown to be equivalent to summing over powers of the dimension with the aforementioned weighting:

$$\sum_{r=1}^{(2n-1)!!} \prod_{k=1}^{n} g_{\mu_{2k-1}\mu_{2k}} \prod_{s=1}^{n} g_{\mu_{\sigma_r(2s-1)}\mu_{\sigma_r(2s)}} = \sum_{k=1}^{n} 2^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} d^k. \tag{D.40}$$

This may be further simplified by noting that

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k = x^{\bar{n}}, \tag{D.41}$$

called the rising factorial or Pochhammer symbol. In our case, we find

$$\sum_{k=1}^{n} 2^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} d^k = 2^n \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \left(\frac{d}{2}\right)^k = 2^n \left(\frac{d}{2}\right)^{\bar{n}}, \tag{D.42}$$

Which is the definition of the Pochhammer k-symbol $(x)_{n,k}$ in the case that $x = d$ and $k = 2$. Note that setting $d = 1$, which is tantamount to ignoring contractions and simply counting our permutations, we have

$$\sum_{k=1}^{n} 2^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} = 2^n \left(\frac{1}{2}\right)^{\bar{n}} = 2^n \frac{\Gamma(n+1/2)}{\Gamma(1/2)} = 2^n \frac{\sqrt{\pi}(2n-1)!!}{2^n} \frac{1}{\sqrt{\pi}} = (2n-1)!!, \tag{D.43}$$

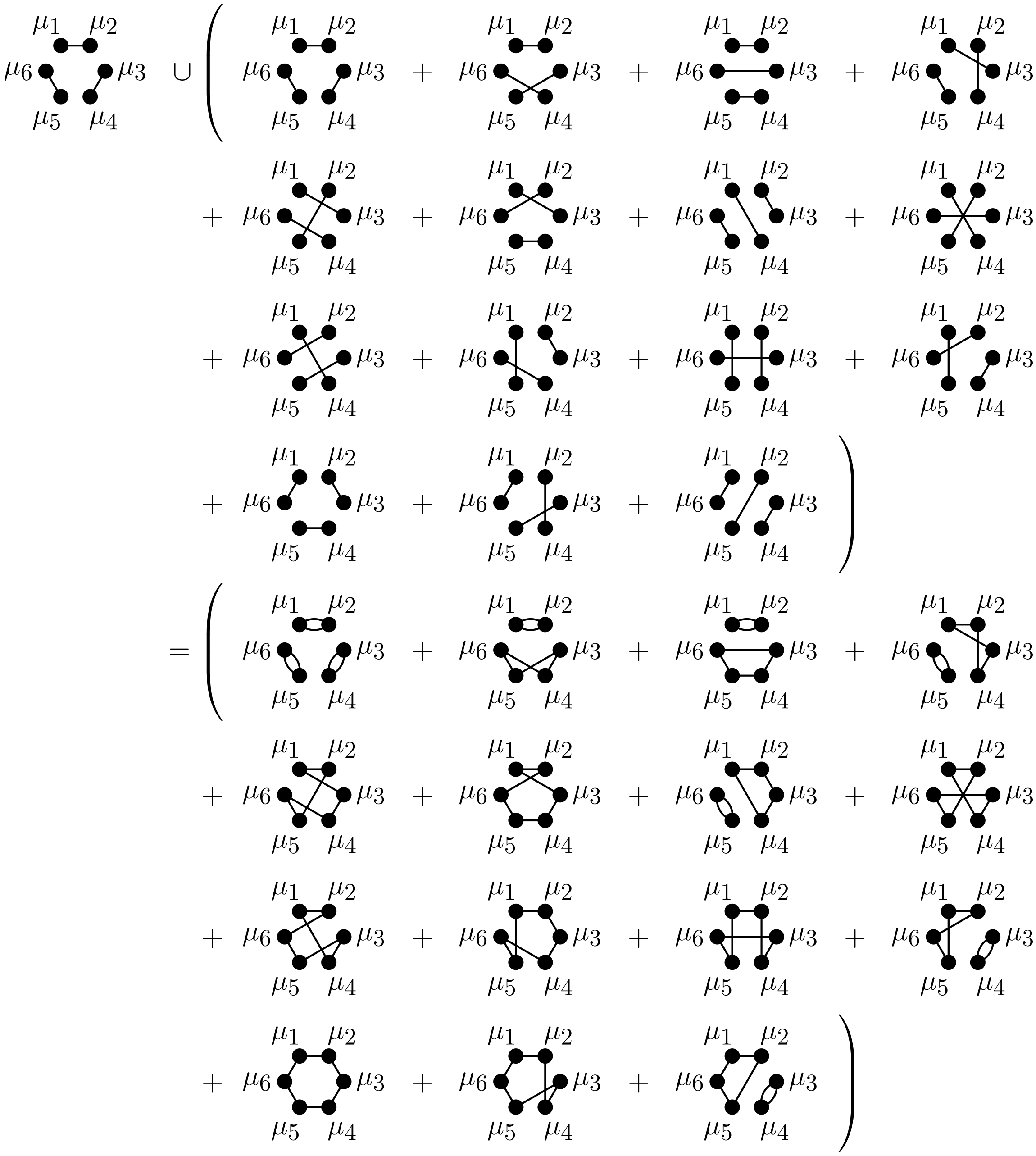
which is exactly the number of ways we may split the set of $2n$ indices into n pairs, which provides a nice sanity check.

We may now solve for our constant of proportionality A:

$$\int_q \frac{e^{-(b+c+z)q^2}}{(q^2)^j} (q^2)^n = (d)_{n,2} A \Rightarrow A = \frac{\Gamma(d/2)}{2^n \Gamma(d/2+n)} \int_q \frac{e^{-(b+c+z)q^2}}{(q^2)^j} (q^2)^n, \tag{D.44}$$

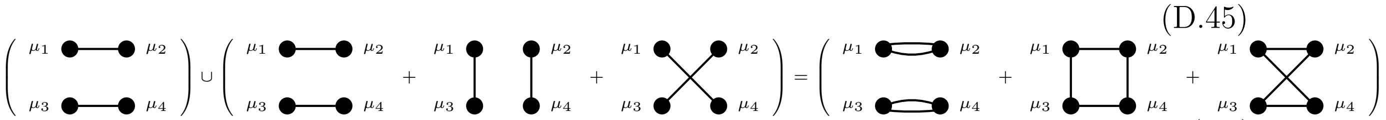
where we have used the the identity $(d)_{n,2} = 2^n \frac{\Gamma(d/2+n)}{\Gamma(d/2)}$. We finally evaluate the momentum integral in its scalar, spherically symmetric form.

D.5 Graphs for $n = 3$



$$\mapsto \; d^3 + d^2 + d^2 + d^2 + d + d + d^2 + d + d + d + d + d^2 + d + d + d^2$$
$$= \; 1 \cdot d^3 + 6 \cdot d^2 + 8 \cdot d$$
$$= \; d(d+2)(d+4).$$

(D.45)



(D.46)

(D.47)

Appendix E

Flowed Beta Function

Appendix F

Gauge Invariance