Student Number: 100

Statistical Mechanics Subject Exam

December 13th, 2019

Do not write your name on the exam.

Possibly useful information:

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}} \text{ (for } a > 0)$$

$$\Gamma(n) = (n-1)! = \int_0^\infty dx x^{n-1} e^{-x}$$

$$\ln N! \approx N \ln N - N \text{ (for } N \gg 1)$$

$$\zeta(m) = \sum_{n=1}^{\infty} n^{-m} = \frac{1}{\Gamma(m)} \int_0^\infty dx \frac{x^{m-1}}{e^x - 1}$$

$$\zeta(m \leq 1) = \infty \quad \zeta(2) = \frac{\pi^2}{6} \quad \zeta(3) = \frac{\pi^4}{90}$$

$$\frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1} dx}{e^x + 1} = \zeta(\nu)[1 - 2^{1-\nu}]$$

$$\int_0^{\pi/2} d\theta \cos\theta \sin\theta = \frac{1}{2}$$

$$\int_0^{\pi/2} d\theta \cos\theta \cos\theta = \frac{\pi}{4}$$

You do not need to explicitly evaluate $\zeta(m)$ or $\Gamma(m)$ if these show up in your solutions.

1. Consider a three-dimensional cubic box containing N indistinguishable particles that can either be in the internal volume of the box or adsorbed onto the surface of the box. The box is immersed in a medium with temperature T. Assume the particles both in the three-dimensional volume and on the two-dimensional surface are non-relativistic and obey Boltzmann statistics (i.e. they behave like an ideal gas). The box has sides of length L and there are N_2 particles on the surface of the box and N_3 in the volume.

   (a) (10 pts) Calculate the canonical partition function for the particles in the volume of the box and the canonical partition function for the particles on the surface of the box.

   (b) (5 pts) For a given T, N, and L, find an expression for the number of particles in the volume. At what temperatures are most particles in the volume?

Extra room for problem 1.

2. (10 pts) Consider a zero temperature gas of N spin-half non-relativistic Fermions in a box of volume V. If there is a hole in the side of the box with area A, what is the average energy of particles escaping from the box? Write your answer in terms of N, V, and the particles mass m (as well as constants).

Extra room for problem 2.

3. (10 pts) Given the D-dimensional Landau-Ginzburg free energy

$$\beta F = \int d^D x \left[t m^2 + u m^6 - m h \right],$$

find the critical exponents β and γ, where $|m| \propto |t|^\beta$ for $t < 0$ and $(\partial m/\partial h)_{h=0} \propto |t|^{-\gamma}$.

Extra room for problem 3.

4. Consider a non-interacting, spin-zero gas of Bosons with the dispersion relation $\epsilon(\vec{p}) = a\sqrt{|\vec{p}|}$, where ϵ is the single-particle energy and a is a constant.

   (a) (10 pts) Find the critical temperature for Bose condensation in three-dimensions in terms of the particle density.

   (b) (5 pts) Is there a lower critical dimension for Bose condensation for this gas? Justify your answer.

Extra room for problem 4.

5. Consider a one-dimensional chain of N coupled oscillators with length L oriented in the x-direction that can move in the y and z directions. In the low-frequency limit, there are two transverse modes that have dispersion relation $\omega = c_s k$ where ω is the wave frequency, k is the wave number, and c_s is the transverse sound speed.

   (a) (5 pts) Calculate the Debye frequency, ω_D for this system.

   (b) (5 pts) What is the specific heat of the system for $T \gg \hbar\omega_D$?

   (c) (10 pts) What is the specific heat for $T \ll \hbar\omega_D$? You do not need to explicitly evaluate any constant integrals, just note they are constant.

Extra room for problem 5.

6. (a) (5 pts) Prove the relation

$$\left(\frac{\partial S}{\partial P}\right)_{T,N} = -\left(\frac{\partial V}{\partial T}\right)_{P,N}.$$

(b) (10 pts) Show that the adiabatic compressibility is related to the isothermal compressibility by the following relation,

$$\kappa_S = \kappa_T - VT\frac{\alpha_P^2}{C_P},$$

where $\kappa_X = -V^{-1}\left(\frac{\partial V}{\partial P}\right)_{X,N}$, $\alpha_P = V^{-1}\left(\frac{\partial V}{\partial T}\right)_{P,N}$, and $C_P = T\left(\frac{\partial S}{\partial T}\right)_{P,N}$.

Extra room for problem 6.

Short Answer Section

7. (4 pts) Consider a system described by the Landau-Ginzburg free energy

$$F = \int d^3x \left[\frac{t}{2}\vec{m}\cdot\vec{m} + u(\vec{m}\cdot\vec{m})^2 + \frac{\kappa}{2}\left(\nabla\vec{m}\right)^2 \right] \quad (1)$$

where $\vec{m}$ is a two-component order parameter. Circle the following statements that are true:

(a) This Hamiltonian possesses a discrete symmetry in m.

(b) This Hamiltonian possesses a continuous symmetry in m.

(c) When this system is taken below it's critical temperature, domain walls will appear.

(d) When this system is taken below it's critical temperature, Goldstone modes will appear.

8. (1 pt) Two systems that belong to the same universality class have the same (circle all that are true)

(a) critical exponents

(b) critical temperature

(c) interparticle interaction

(d) order parameter

(e) symmetries of the order parameter

9. (2 pts) A solid, impermeable container, which has two chambers separated by a impermeable but flexible membrane, is placed in a room with temperature T and allowed to equilibrate. If the entropy of the gas can be written as $S(E, N_A, N_B, V_{\text{tot}}, X)$ where $V_{\text{tot}} = V_A + V_B$ and $X = V_A/V_{\text{tot}}$, which of the following is/are true? (V_A and V_B are the volumes of the two chambers in the box)

(a) The total entropy of the box and room is maximized.

(b) The total Helmholtz free energy of the gas is minimized.

(c) The total Helmholtz free energy of the gas is maximized.

(d) The total Gibbs free energy of the gas is minimized.

(e) The total Gibbs free energy of the gas is maximized.

(f) The total Grand potential of the gas is minimized.

(g) The total Grand potential of the gas is maximized.

10. Consider a system containing two indistinguishable spin-1 Bosons with three spin-independent single-particle energy levels $-\epsilon$, 0, and ϵ that is thermalized at temperature T. (2 pts each)

(a) What is the average total energy when $T = 0$?

____

(b) What is the entropy when $T = 0$?

____

(c) What is the average total energy when $T \to \infty$?

____

(d) What is the entropy when $T \to \infty$?

____