Statistical Mechanics Subject Exam, Wednesday, Dec. 14, 2016

SECRET STUDENT NUMBER: STUDNUMBER

DO NOT WRITE YOUR NAME OR STUDENT NUMBER ON ANY SHEET!

FUN FACTS TO KNOW AND TELL

$$\int_0^\infty dx\ \frac{x^{n-1}}{e^x-1} = \Gamma(n)\zeta(n), \qquad \int_0^\infty dx\ \frac{x^{n-1}}{e^x+1} = \Gamma(n)\zeta(n)\left[1-(1/2)^{n-1}\right],$$

$$\zeta(n) \equiv \sum_{m=1}^{\infty} m^{-n}, \quad \Gamma(n) \equiv (n-1)!,$$

$$\zeta(3/2) = 2.612375..., \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(3) = 1.20205..., \quad \zeta(4) = \frac{\pi^4}{90},$$

$$\int_{-\infty}^{\infty} dx\ e^{-x^2/2} = \sqrt{2\pi}, \qquad \int_0^\infty dx\ x^n e^{-x} = n!$$

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LONG ANSWER SECTION

1. Consider a large two-dimensional array of N coupled harmonic oscillators in area A that lie in the $x-y$ plane when at rest. The oscillator's movement is also confined to the $x-y$ plane. In the absence of the coupling, each oscillator has a fundamental frequency ω_0. After coupling both the longitudinal and transverse sound modes have a speed c_s.

   (a) (10 pts) Solve for the Debye frequency, ω_D, in terms of ω_0, N, A and c_s.

   (b) (10 pts) For $T << \hbar\omega_D$, find the specific heat per unit area,

   $$C \equiv (1/A)d\langle E\rangle/dT.$$

   (c) (5 pts) What is $C(T \to \infty)$?

Extra workspace for #1

2. (10 pts) N ink molecules are placed in a vessel of length L, $0 < x < L$. The molecules diffuse according to a diffusion constant D, i.e., the density satisfies the diffusion equation,

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2}.$$

At $t = 0$, the density has the form

$$\rho(x, t = 0) = A \sin(\pi x / L),$$

Assuming the molecules **adsorb** at the boundaries (stick), find $\rho(x, t > 0)$.

3. A two-dimensional gas of non-relativistic spin-1/2 fermions of mass m are confined within an area A and are thermalized according to a chemical potential μ and a temperature T. Originally, the temperature is $T = 0$ and the chemical potential is $\mu > 0$.

   (a) (5 pts) In terms of m and A, find the density of single-particle states, $D(\epsilon)$.

   (b) (5 pts) In terms of A, m and μ, find the average number of particles when $T = 0$.

   (c) (10 pts) Assuming μ is held constant while the temperature is raised slightly, find the change in the average number of particles to second order in the temperature. Express your answer in the form: (find b_1 and b_2)

$$N = N_0 + b_1 T + b_2 T^2.$$

Extra work space for #3

4. Suppose the average energy $\bar{E}$ and the average number of particles $\bar{N}$ in a one-dimensional system of extent L are given as a function of T, L and $\alpha \equiv -\mu/T$. Further assume that L is much larger than any microscopic scale or correlation length of the system.

   (a) (10 pts) Derive an expression for the specific heat per unit length,

   $$C \equiv \frac{1}{L} \frac{\partial \bar{E}}{\partial T}\bigg|_N ,$$

   in terms of $T, L, \bar{E}, \bar{N}, \partial_T \bar{E}|_\alpha, \partial_\alpha \bar{E}|_T$, $\partial_T \bar{N}|_\alpha$ and $\partial_\alpha \bar{N}|_T$.

   (b) (10 pts) Assume the correlations in the system are sufficiently local they can be expressed in terms of delta functions,

   $$\begin{aligned} \langle \Delta\rho(0)\Delta\rho(x)\rangle|_{\alpha,T} &= A_{\rho\rho}\delta(x), \\ \langle \Delta\epsilon(0)\Delta\epsilon(x)\rangle|_{\alpha,T} &= A_{\epsilon\epsilon}\delta(x), \\ \langle \Delta\epsilon(0)\Delta\rho(x)\rangle|_{\alpha,T} &= A_{\epsilon\rho}\delta(x), \end{aligned}$$

   where ϵ and ρ are the energy density and number density respectively. Express C in terms of T, α, $\bar{N}$, $\bar{E}$, $A_{\rho\rho}$, $A_{\epsilon\epsilon}$ and $A_{\rho\epsilon}$.

Extra work space for #4

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SHORT ANSWER SECTION

5. (2 pts each) Three identical spin-zero particles can each occupy one of two single-particle energy levels, 0 and ϵ.

   (a) What is the average energy when $T = 0$? ____________

   (b) What is the entropy when $T = 0$? ____________

   (c) What is the average energy when $T >> \epsilon$? ____________

   (d) What is the entropy when $T >> \epsilon$? ____________

6. (3 pts) If you read an article where the authors maximize the pressure to solve for an order parameter ϕ, which quantities can you assume were fixed as ϕ was varied? Circle all that are true.

   (a) entropy

   (b) temperature

   (c) particle number

   (d) density

   (e) chemical potential

   (f) energy density

7. (4 pts) A box of volume $2V$ is initially set up with N indistinguishable particles partitioned on one side and N of the same species of particles on the right side. Both sides initially have volume V, are at the same temperature T, and behave like ideal gases. The partition is an excellent conductor of heat so heat can readily flow from one partition to the other, but the overall system is extremely well insulated from the outside, and no heat leaves the system. Beginning at $t = 0$, the partition is slowly (compared to rate at which heat moves from one side to the other) moved until the left side has volume $3V/2$ and the right side has volume $V/2$. Circle all that are true:

   - The pressures remain equal to one another, $P_L = P_R$
   - The temperatures remain equal to one another, $T_L = T_R$
   - The entropy densities remain equal to one another, $s_L = s_R$
   - The overall entropy remains constant, $S_L + S_R$ =constant
   - The overall energy remains constant, $E_L + E_R$ =constant

8. (3 pts) A system is set up in a large volume V at temperature T and average density $\bar{\rho}$. The matter has a critical temperature T_c for a liquid-gas phase transition, and is well described by a Van der Waals equation of state. Circle all that are true.

   - If $T < T_c$ the system must be undergoing phase separation.
   - If two phases are in coexistence, the two regions have equal entropy densities.
   - If two phases are in coexistence, the two regions have equal energy densities.

9. (3 pts) Two phase transitions of the same universality class have: Circle all that are true

   - The same order parameters
   - The same critical exponents
   - The same Goldstone bosons
   - The same dimensionality
   - The same symmetries

10. (3 pts) An ink molecule is placed at a position $x = y = z = 0$ at time $t = 0$ in a liquid. The molecule is initially given a thermal (random) velocity. Laura models the motion of the molecule with a Langevin equation and Darko models the motion with a diffusion equation. For each question about the spread of positions $\langle r^2 \rangle = \langle x^2 + y^2 + z^2 \rangle$, fill in the blank with either *"significantly greater than 1"*, *"significantly less than 1"*, or *"close to 1"*

    (a) For short (but non-zero) times,

    $$\frac{\langle r^2 \rangle_{\rm Laura}}{\langle r^2 \rangle_{\rm Darko}} \text{ is } \underline{\hspace{4cm}}.$$

    (b) For long times,

    $$\frac{\langle r^2 \rangle_{\rm Laura}}{\langle r^2 \rangle_{\rm Darko}} \text{ is } \underline{\hspace{4cm}}.$$

11. (4 pts) Gas A consists of N spin-zero particles of mass m in volume V at temperature T. Gas B has particles of the same mass, at the same temperature and confined in the same volume. However, gas B consists of spin-1/2 particles, all of which have their spin pointing upward. For each question answer A, B or "*neither*".

    - Which gas has higher pressure? ________________________
    - Which gas has higher chemical potential? ________________________

12. (2 pts) Bob's universe expands without acceleration, with collective velocity $\vec{v} = \vec{r}/t$. Bob's universe is made up of massive non-relativistic particles at time τ_0 when the temperature is T_0 at every point. After τ_0 the universe expands hydrodynamically until time τ_f. The temperature falls with a power law, $T_f = T_0(\tau_0/\tau_f)^n$. What is n?

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