PHY-841: Classical Electrodynamics / Subject Exam / May 1, 2019

SECRET STUDENT NUMBER: STUDNUMBER

Please read all of the following before starting the exam:

- Do not write your name or student PID on any page of the exam. If you require extra paper, write the secret student number and the relevant problem number on the extra pages.
- You may use a simple calculator, but no phone, networking device, external notes, books, etc.
- All problems are in S.I. units. Please give your answers in terms of the given variables and use S.I. units, too.
- A complete answer usually includes a derivation of the result (unless stated otherwise). Show all work as neatly and logically as possible to maximize your credit. State clearly which equations were used. Circle or otherwise indicate your final answers.
- This exam has 12 questions for a total of 100 points. Please make sure that you have all of the pages.
- **Good luck!**

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\begin{aligned}
\vec{a}\times(\vec{b}\times\vec{c}) &= \vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b}),\\
\vec{a}\cdot(\vec{b}\times\vec{c}) &= \vec{b}\cdot(\vec{c}\times\vec{a})=\vec{c}\cdot(\vec{a}\times\vec{b}),\\
(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) &= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c}),\\
\vec{\nabla}\times(\vec{\nabla}\psi) &= 0,\\
\vec{\nabla}\cdot(\vec{\nabla}\times\vec{a}) &= 0,\\
\vec{\nabla}\times(\vec{\nabla}\times\vec{a}) &= \vec{\nabla}(\vec{\nabla}\cdot\vec{a})-\nabla^2\vec{a},\\
\vec{\nabla}\cdot(\psi\vec{a}) &= \vec{a}\cdot\vec{\nabla}\psi+\psi\vec{\nabla}\cdot\vec{a},\\
\vec{\nabla}\times(\psi\vec{a}) &= \vec{\nabla}\psi\times\vec{a}+\psi\vec{\nabla}\times\vec{a},\\
\vec{\nabla}(\vec{a}\cdot\vec{b}) &= (\vec{a}\cdot\vec{\nabla})\vec{b}+(\vec{b}\cdot\vec{\nabla})\vec{a}+\vec{a}\times(\vec{\nabla}\times\vec{b})+\vec{b}\times(\vec{\nabla}\times\vec{a}),\\
\vec{\nabla}\cdot(\vec{a}\times\vec{b}) &= \vec{b}\cdot(\vec{\nabla}\times\vec{a})-\vec{a}\cdot(\vec{\nabla}\times\vec{b}),\\
\vec{\nabla}\times(\vec{a}\times\vec{b}) &= \vec{a}(\vec{\nabla}\times\vec{b})-\vec{b}(\vec{\nabla}\times\vec{a})+(\vec{b}\cdot\vec{\nabla})\vec{a}-(\vec{a}\cdot\vec{\nabla})\vec{b},\\
\vec{\nabla}\cdot\vec{r} &= 3,\\
\vec{\nabla}\times\vec{r} &= 0,\\
\vec{\nabla}\cdot\hat{r} &= 2/r,\\
\vec{\nabla}\times\hat{r} &= 0,\\
\vec{\nabla}r &= \hat{r},\\
\vec{\nabla}\frac{1}{r} &= -\frac{\hat{r}}{r^2},\\
\vec{\nabla}\cdot(\hat{r}f(r)) &= \frac{2}{r}f+\frac{df}{dr},\\
(\vec{a}\cdot\vec{\nabla})\hat{r} &= \frac{1}{r}[\vec{a}-\hat{r}(\vec{a}\cdot\hat{r})]=\frac{\vec{a}_\perp}{r},\\
\vec{\nabla}^2\left(\frac{1}{r}\right) &= -4\pi\delta(\vec{r}),\\
\int_V d^3r\ \vec{\nabla}\cdot\vec{A} &= \int_S d\vec{S}\cdot\vec{A},\\
\int_V d^3r\ \vec{\nabla}\psi &= \int_S \psi\ d\vec{S},\\
\int_V d^3r\ \vec{\nabla}\times\vec{A} &= \int_S d\vec{S}\times\vec{A},\\
\int_V d^3r\ (\phi\nabla^2\psi+\vec{\nabla}\phi\cdot\vec{\nabla}\psi) &= \int_S \phi\ d\vec{S}\cdot\vec{\nabla}\psi,\\
\int_V d^3r\ (\phi\nabla^2\psi-\psi\nabla^2\phi) &= \int_S (\phi\vec{\nabla}\psi-\psi\vec{\nabla}\phi)\cdot d\vec{S},\\
\int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{S} &= \oint d\vec{\ell}\cdot\vec{A},\\
\int_S d\vec{S}\times\vec{\nabla}\psi &= \oint_C d\vec{\ell}\ \psi,\\
\int_0^{2\pi}\sin^2(\alpha)\,d\alpha &= \pi\,,\\
\int_0^{\pi}\sin^3(\alpha)\,d\alpha &= \frac{4}{3}\,.
\end{aligned}
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\begin{aligned}
\mathscr{L} &= -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu \\
L &= \frac{1}{\gamma}(-mc^2 - qA_\mu u^\mu) \\
\frac{dp^\mu}{d\tau} &= qF^{\mu\nu}u_\nu \\
\partial_\mu F^{\mu\nu} &= \mu_0 J^\nu \\
\partial_\mu \tilde{F}^{\mu\nu} &= 0 \\
\partial_\mu J^\mu &= 0 \\
(g_{\mu\nu}) &= \mathrm{diag}(1,-1,-1,-1), \\
\vec{\beta} &= \vec{v}/c, \\
\gamma &= 1/\sqrt{1-\beta^2}, \\
x'^\mu &= \Lambda^\mu{}_\nu x^\nu \\
(\Lambda^\mu{}_\nu) &= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\end{aligned}
$$

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\begin{aligned}
x^\mu &= (ct, x, y, z), \\
\partial^\mu &= ((1/c)\partial/\partial t, -\vec{\nabla}) \\
k^\mu &= (\omega/c, \vec{k}), \\
u^\mu &= (\gamma c, \gamma\vec{v}), \\
p^\mu &= (E/c, \vec{p}), \\
A^\mu &= (\phi/c, \vec{A}), \\
J^\mu &= (c\rho, \vec{j}), \\
F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu, \\
\tilde{F}^{\mu\nu} &= \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}
\end{aligned}
$$

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\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho, \qquad \vec{\nabla}\cdot\vec{B} = 0,
$$

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\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{j}, \qquad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0,
$$

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\begin{aligned}
\frac{d\vec{p}}{dt} &= q(\vec{E} + \vec{v}\times\vec{B}) \\
\vec{E} &= -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} \\
\vec{B} &= \vec{\nabla}\times\vec{A}
\end{aligned}
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\begin{aligned}
&(\vec{E}_{\mathrm{out}} - \vec{E}_{\mathrm{in}})\cdot\hat{n} = \sigma/\epsilon_0 \\
&\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V d^3r'\,\rho(\vec{r}\,')G(\vec{r},\vec{r}\,') - \frac{1}{4\pi}\int_S dA'\,\phi(\vec{r}\,')\frac{\partial G(\vec{r},\vec{r}\,')}{\partial n'} \\
&\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V d^3r'\,\rho(\vec{r}\,')G(\vec{r},\vec{r}\,') - \frac{1}{4\pi}\int_S dA'\,\frac{\partial\phi(\vec{r}\,')}{\partial n'}G(\vec{r},\vec{r}\,') + \langle\phi\rangle_S \\
&\quad \phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}r^l + B_{lm}/r^{l+1}\right]Y_{lm}(\theta,\varphi) \\
&\quad \phi = \sum_{l=0}^{\infty}\left[a_{lm}r^l + b_{lm}/r^{l+1}\right]P_l(\cos\theta) \\
&\quad \phi = a_0 + b_0\varphi + (c_0 + d_0\varphi)\ln\rho + \sum_{n=1}^{\infty}\Big[\rho^{\nu_n}(a_n\cos\nu_n\varphi + b_n\sin\nu_n\varphi) \\
&\qquad\qquad + \rho^{-\nu_n}(c_n\cos\nu_n\varphi + d_n\sin\nu_n\varphi)\Big]
\end{aligned}
$$

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\begin{aligned}
P_l^m(x) &= \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l \\
P_l(1) &= 1 \\
Y_{lm}(\theta,\varphi) &= \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}e^{im\varphi}P_l^m(\cos\theta) \\
Y_{0,0} &= \frac{1}{\sqrt{4\pi}} \\
Y_{1,0} &= \sqrt{\frac{3}{4\pi}}\cos\theta \\
Y_{1,\pm1} &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi} \\
Y_{2,0} &= \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta-1\right) \\
Y_{2,\pm1} &= \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\varphi} \\
Y_{2,\pm2} &= \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\varphi}
\end{aligned}
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\begin{aligned}
\phi &= \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{4\pi}{2l+1}\right)\frac{q_{lm}}{r^{l+1}}Y_{lm}(\theta,\varphi) \\
q_{lm} &= \int d^3r'\,\rho(\vec{r}')r'^l Y_{lm}^*(\theta,\varphi)
\end{aligned}
$$

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\begin{aligned}
\phi &= \frac{1}{4\pi\epsilon_0}\left(\frac{Q_{\text{tot}}}{r}+\frac{p_i\hat{r}_i}{r^2}+\frac{1}{2!}Q_{ij}\frac{\hat{r}_i\hat{r}_j}{r^3}+\dots\right) \\
p_i &= \int d^3\,r'\rho(\vec{r}')\,r_i' \\
Q_{ij} &= \int d^3r'\,\rho(\vec{r}')\,(3r_i'r_j'-\delta_{ij}r'^2) \\
U &= Q_{\text{tot}}\phi(\vec{r})-p_iE_i(\vec{r})-\frac{1}{6}Q_{ij}\partial_iE_j+\dots
\end{aligned}
$$

$$
\begin{aligned}
\vec{A}(\vec{r}) &= \frac{\mu_0}{4\pi}\left(\frac{\vec{m}\times\vec{r}}{r^3}+\dots\right) \\
\vec{B}(\vec{r}) &= \frac{\mu_0}{4\pi}\left(\frac{3(\vec{m}\cdot\hat{r})\hat{r}-\vec{m}}{r^3}+\dots\right) \\
\vec{m} &= \frac{1}{2}\int d^3r'\,\vec{r}'\times\vec{j}(\vec{r}') \\
U &= -m_iB_i(\vec{r})+\dots \\
\vec{\tau} &= \vec{m}\times\vec{B}+\vec{r}\times\vec{F}+\dots \\
\vec{m} &= \frac{q}{2m}\vec{L}
\end{aligned}
$$

$$
\begin{aligned}
\vec{A} &= \frac{\mu_0}{4\pi}\left[\frac{1}{r}\left(\dot{\vec{p}}\right) + \frac{1}{rc}\left(\dot{\vec{m}}\times\hat{r} + \frac{1}{6}\ddot{\vec{Q}}\right) + \ldots\right] \\
\vec{B} &= \frac{\mu_0}{4\pi c}\left[\frac{1}{r}\left(\ddot{\vec{p}}\times\hat{r}\right) + \frac{1}{rc}\left((\ddot{\vec{m}}\times\hat{r})\times\hat{r} + \frac{1}{6}\dddot{\vec{Q}}\times\hat{r}\right) + \ldots\right] \\
\vec{E} &= \frac{\mu_0}{4\pi}\left[\frac{1}{r}\left((\ddot{\vec{p}}\times\hat{r})\times\hat{r}\right) + \frac{1}{rc}\left(\ddot{\vec{m}}\times\hat{r} + \frac{1}{6}(\dddot{\vec{Q}}\times\hat{r})\times\hat{r}\right) + \ldots\right] \\
\frac{dP}{d\Omega} &= |\vec{S}|r^2 = \frac{c}{\mu_0}|\vec{B}|^2 r^2 \\
P(t) &= \frac{\mu_0}{4\pi c}\left(\frac{2}{3}|\ddot{\vec{p}}|^2 + \frac{2}{3c^2}|\ddot{\vec{m}}|^2 + \frac{1}{180c^2}\dddot{Q}_{ij}\dddot{Q}_{ji}\right) \\
\vec{B} &= i\vec{k}\times\vec{A} \\
\vec{E} &= c\vec{B}\times\hat{r} \\
\vec{E}'_{\parallel} &= \vec{E}_{\parallel}, \qquad \vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + c\vec{\beta}\times\vec{B}) \\
\vec{B}'_{\parallel} &= \vec{B}_{\parallel}, \qquad \vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - \frac{1}{c}\vec{\beta}\times\vec{E}) \\
u &= \frac{1}{2}\left(\epsilon_0|\vec{E}|^2 + \frac{1}{\mu_0}|\vec{B}|^2\right) \\
\vec{S} &= \frac{1}{\mu_0}\vec{E}\times\vec{B} \\
0 &= \vec{E}\cdot\vec{j} + \frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} \\
\vec{g} &= \epsilon_0\vec{E}\times\vec{B} \\
\sigma_{ij} &= \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}|\vec{E}|^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}|\vec{B}|^2\right) \\
0 &= \left(\rho\vec{E} + \vec{j}\times\vec{B} + \frac{\partial}{\partial t}\vec{g}\right)_j - \frac{\partial}{\partial x_i}\sigma_{ij}
\end{aligned}
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LONG ANSWER SECTION

Charges Observed at Large Distance

Consider three point charges: one charge $+q$ at position $(x, y, z) = (0, 0, -a)$, one charge $+q$ at position $(0, 0, a)$ and one charge $+5q$ at position $(0, 0, 0)$ where $a > 0$. The potential of these charges is observed at a point $\vec{r}$ far away from the origin, $r = |\vec{r}| \gg a$. Consider the $1/r$ expansion of the potential in terms of Cartesian multipoles.

1. [15 pts] Determine the Cartesian monopole, dipole and quadrupole moments. Make sure to specify all components, including zero components.

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Conducting Sphere in Electric Field

Consider an electrostatic problem, where a perfectly conducting sphere of radius R is brought into an external electric field $\vec{E} = E_0\hat{z}$. The original electric field remains approximately unmodified far away from the sphere. The sphere is centered at the origin and held at a constant potential ϕ_0.

2. [15 pts] Calculate the potential outside of the sphere.

Rotating Cylinder

A solid cylinder of radius R and height h is uniformly charged over its volume with a total charge of $q > 0$. It's total mass m is also uniformly spread over its volume. The cylinder is rotating around its cylindrical axis with a small angular velocity ω.

3. [10 pts] Calculate the magnetic moment of the cylinder around its rotation axis.

4. [5 pts] The cylinder is placed in a constant uniform magnetic field $\vec{B} = B\hat{z}$. In which direction should the angular velocity vector $\vec{\omega}$ of the cylinder point in order to maximize the magnitude of the torque on the cylinder ? What is the effect of the torque on the motion of the cylinder ?

Moving Long Cylinder

Consider a uniformly charged, non-conducting, very long cylinder with radius R and charge density ρ_0. The cylinder is aligned with the z axis and moves in the positive z direction.

5. [10 pts] In the rest frame F' of the cylinder, what are the values of the electric field $\vec{E}'(\vec{r}')$ and the magnetic field $\vec{B}'(\vec{r}')$ outside of the cylinder ?

6. [10 pts] In the lab frame F, the cylinder moves with velocity $\vec{v} = v\hat{z}$. In this frame, what are the values of the electric field $\vec{E}(\vec{r})$ and the magnetic field $\vec{B}(\vec{r})$ outside of the cylinder ?

See-Saw Antenna

Consider a linear antenna of length d with a narrow gap in the center for the purposes of excitation. Assume a hypothetical current which oscillates harmonically with time. For a given point in time, the current is in the same direction for both halves of the antenna, having a value of I_0 at the gap and falling linearly to zero at the ends. For an angular frequency of the excitation $\omega \ll c/d$, consider the radiation at a large distance r from the antenna. The antenna is aligned with the z axis and centered at the origin.

7. [10 pts] What is the time averaged total power radiated off the antenna ?

8. [5 pts] In order to receive a strong signal, at which position should one place the receiver: $\vec{a} = (1, 0, 0)\,\mathrm{r}$, $\vec{b} = (0, 0, \sqrt{5})\,\mathrm{r}$, or $\vec{c} = (1, 0, 1)\,\mathrm{r}$? Assume the distance r to be much larger than d. Describe how you arrive at your conclusion.

9. [5 pts] Compare the linear antenna to a circular antenna, where the current oscillates harmonically with time, but is constant along the wire for a given point in time. Assume the length of the wire to be of order d. In the limit $\omega d/c \to 0$ for fixed sizes of the antennas, which of the two radiates with larger power ? Describe *why*.

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SHORT ANSWER SECTION

10. [5 pts] Consider a plane wave electric field given by $E_x = E_1 \cos(kz - \omega t)$, $E_y = E_2 \cos(kz - \omega t - \alpha)$, $E_z = 0$, where $E_1 > 0$, $E_2 \geq 0$ and $0 \leq \alpha \leq 2\pi$ are three real constants, and $\omega = kc$. For a given E_1, list the possible values of E_2 and α for (i) linear polarization and (ii) right-handed circular polarization.

11. [5 pts] Consider a perfect conducting sphere with radius R centered at the origin. At the point $(0, 0, R)$, the electric field outside of the conductor has magnitude E. What is the surface charge density in that point ?

12. [5 pts] The lifetime of a muon in rest frame is $2\mu s$. A muon is produced and travels with a speed of $0.8c$ in the laboratory frame. How far will it travel on average before it decays ?